NOTES ON COTORSION THEORIES AND MODEL CATEGORIES

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Introduction

These are notes for two talks given by Mark Hovey at the Summer School on the Interactions between Homotopy Theory and Algebra at the University of Chicago, July 26 to August 6, 2004. Because they are notes, they are a bit more chatty and a bit more likely to contain errors than a paper would be, so *caveat lector*. They are based on the papers [Hov02], [Gil04b], and [Gil04a], and concern the relationship between Quillen model structures on abelian categories and cotorsion theories.

1. Cotorsion theories

Cotorsion theories were invented by Luigi Salce [Sal79] in the category of abelian groups, and were rediscovered by Ed Enochs and coauthors in the 1990's. A **cotorsion theory** in an abelian category $\mathcal{A}$ is a pair $(\mathcal{D}, \mathcal{E})$ of classes of objects of $\mathcal{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. That is, we have

(1) $D \in \mathcal{D}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $E \in \mathcal{E}$; and
(2) $E \in \mathcal{E}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $D \in \mathcal{D}$.

The most obvious example of a cotorsion theory is when $\mathcal{D} = \mathcal{A}$, in which case $\mathcal{E}$ is the class of injective objects. Similarly, we could let $\mathcal{E}$ be everything, in which case $\mathcal{D}$ is the class of projective objects.

Based on this example, we say that a cotorsion theory $(\mathcal{D}, \mathcal{E})$ **has enough projectives** if for all X in our abelian category $\mathcal{A}$ there is a short exact sequence

$$0 \to E \to D \to X \to 0$$

where $D \in \mathcal{D}$ and $E \in \mathcal{E}$. So $\mathcal{A}$ has enough projectives in the usual sense if and only if the cotorsion theory (projectives, everything) has enough projectives. On the other hand, the cotorsion theory (everything, injectives) always has enough projectives. Dually, we say that

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$(\mathcal{D}, \mathcal{E})$ **has enough injectives** if for all X in $\mathcal{A}$ there is a short exact sequence

$$0 \to X \to E \to D \to 0$$

with $E \in \mathcal{E}$ and $D \in \mathcal{D}$. If $(\mathcal{D}, \mathcal{E})$ has enough projectives and enough injectives, we say that it is a **complete cotorsion theory**.

Perhaps the most useful cotorsion theory, and the one that gives the subject its name, is the flat cotorsion theory. Here $\mathcal{A}$ is the category of R-modules for some ring R, $\mathcal{D}$ is the category of flat R-modules, and $\mathcal{E}$ is what it has to be, the collection of all modules E such that $\operatorname{Ext}^1_R(D, E) = 0$ for all flat D. Such modules are called cotorsion modules.

It is not at all obvious that this is a cotorsion theory, or what cotorsion modules look like. A brief digression may be warranted to describe this important example.

First of all, a short exact sequence

$$0 \to A \to B \to C \to 0$$

is called **pure** if it remains exact upon apply the functor $M \otimes_R (-)$ for any R-module M. My favorite reference for purity and many other algebraic topics is [Lam99]; purity is discussed in Section 4J, where it is proved, among other things, that the pure exact sequences are the colimits of split exact sequences, and that any short exact sequence where the right-hand entry C is flat is automatically pure. Purity is of considerable interest to logicians interested in the model theory of modules [Her97].

A module A is **pure injective** if every pure exact sequence with A as the left-hand entry is in fact split. There are lots of these around; most importantly, if M is any R-module, then $M^+ = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is always pure injective. (Note that if M is a left R-module, then M^+ is a right R-module). And every pure injective module is cotorsion (because any short exact sequence that ends in a flat is automatically pure), so this gives us a source of cotorsion modules. Using these facts, we can prove that (flats, cotorsions) is in fact a cotorsion theory. Indeed, it suffices to show that if $\operatorname{Ext}^1(D, E) = 0$ for all cotorsion E, then D is flat. But this means that $\operatorname{Ext}^1(D, M^+) = 0$ for all M. Using the derived version of the Hom and tensor adjointness, and the fact that $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, we see that $\operatorname{Tor}^1(D, M)^+ = 0$ for all M, which implies that D is flat.

It was an open question for a long time whether the cotorsion theory (flats, cotorsions) was complete. This became known as the **flat cover conjecture**. It was eventually proved when some logicians (Eklof-Trlifaj [ET01]) rediscovered some model category theoretic techniques (though I don't believe they realized the connection). Bican, El Bashir, and Enochs then used the Eklof-Trlifaj result to prove the flat cover conjecture [BEBE01].

2. Relation between cotorsion theories and model categories

Suppose we have a cotorsion theory $(\mathcal{D}, \mathcal{E})$. This means that given any short exact sequence

$$0 \to A \xrightarrow{i} B \to D \to 0$$

with $D \in \mathcal{D}$, and for any $E \in \mathcal{E}$, the map

$$\mathcal{A}(B, E) \to \mathcal{A}(A, E)$$

is surjective. That is, if we imagine i as a trivial cofibration, $\mathcal{E}$ consists of fibrant objects. This suggests that there should be some relation between model categories and cotorsion theories.

2.1. **Abelian model categories.** For this to be true, we clearly need some relation between the model structure on $\mathcal{A}$ and the abelian structure.

Definition 2.1. An **abelian model category** is an abelian category $\mathcal{A}$ equipped with a model structure such that

(1) Every cofibration is a monomorphism.
(2) A map is a (trivial)fibration if and only if it is an epimorphism with (trivially)fibrant kernel.

Most of the standard model structures on abelian categories are abelian model structures. For example, in the projective model structure on chain complexes, the cofibrations are the monomorphisms with cofibrant (=DG-projective) cokernel, the fibrations are the epimorphisms, and the trivial fibrations are the epimorphisms with exact kernel.

A trivial example of a model structure that is not abelian is the one where weak equivalences are isomorphisms and all maps are cofibrations and fibrations. A less trivial example is the absolute model structure on chain complexes, where the weak equivalences are chain homotopy equivalences, and everything is cofibrant and fibrant. In this model structure, the cofibrations are the degreewise split monomorphisms and the fibrations are the degreewise split epimorphisms. So an epimorphism with fibrant kernel is usually not a fibration. However, it is possible to modify the definition of abelian model category to include this example and many others, using the idea if a proper class of short exact sequences. This is the same thing as a subbufunctor of the Ext functor, so there is also a modified definition of a cotorsion theory using this subfunctor. In the case of the absolute model structure, our proper class is the class of degreewise split sequences.

Another point is that the definition of abelian model category seems so asymmetric. Why should we not also require that a map be a (trivial) cofibration if and only if it is a monomorphism with (trivially) cofibrant cokernel? The answer is: because it follows anyway (proof omitted).

Now does an abelian model structure have something to do with cotorsion theories? Yes!

Proposition 2.2. *Suppose $\mathcal{A}$ is an abelian model category. Let $\mathcal{C}$ denote the class of cofibrant objects, $\mathcal{F}$ the class of fibrant objects, and $\mathcal{W}$ the class of trivial objects (those that are weakly equivalent to 0). Then $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ and $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ are complete cotorsion theories.*

Proof. Just do the $(\mathcal{C},\mathcal{F}\cap\mathcal{W})$ case as the other is similar. There are 5 steps to the argument.

(1) $\operatorname{Ext}^1(C,K)=0$ for cofibrant C and trivially fibrant K. Prove this by realizing an element of $\operatorname{Ext}^1(C,K)$ as a short exact sequence and lifting, using the fact that C is cofibrant and a map is a trivial fibration when it is an epimorphism with trivially fibrant kernel.
(2) If $\operatorname{Ext}^1(A,K)=0$ for all trivially fibrant K, then A is cofibrant. Prove this by showing that $\mathcal{A}(A,-)$ takes trivial fibrations to surjections, so $0\to A$ has the left lifting property with respect to trivial fibrations.
(3) If $\operatorname{Ext}^1(C,X)=0$ for all cofibrant C, then X is trivially fibrant. Prove this by showing $X\to 0$ has the right lifting property with respect to cofibrations.
(4) The cotorsion theory has enough projectives. Prove this by factoring $0\to X$ into a cofibration followed by a trivial fibration.
(5) The cotorsion theory has enough injectives. Prove this by factoring $X\to 0$ into a cofibration followed by a trivial fibration.

□

2.2. **From cotorsion theories to an abelian model category.** We have seen that an abelian model structure gives rise to two compatible complete cotorsion theories. Can we go the other way? Well, no, not without some more hypotheses. Recalling the model category axioms, there is not just lifting and factorization; there is also two out of three and retracts. To make this work we are going to need some hypothesis on $\mathcal{W}$.

Definition 2.3. A nonempty subcategory of an abelian category is called **thick** if it is closed under retracts and whenever two out of three entries in a short exact sequence are in the thick subcategory, so is the third.

Lemma 2.4. *Suppose $\mathcal{A}$ is an abelian model category and $\mathcal{W}$ is the class of trivial objects. Then $\mathcal{W}$ is thick.*

We leave the proof to the reader.

So now we get the desired theorem.

Theorem 2.5. *Suppose $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{W}$ are three classes of objects in a bicomplete abelian category $\mathcal{A}$, such that*

(1) *$\mathcal{W}$ is thick.*

(2) *$(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion theories.*

Then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects, and $\mathcal{W}$ is the class of trivial objects.

The proof of this theorem is interesting, as it does not follow the usual path for proving something is a model category. Usually the main difficulty is proving the lifting and factorization axioms, but in this case the main difficulty is defining the weak equivalences and proving the two-out-of-three axiom, which is usually trivial.

It is clear that we should define f to be a cofibration if f is a monomorphism with cokernel in $\mathcal{C}$, a fibration if f is an epimorphism with kernel in $\mathcal{F}$, a trivial cofibration if f is a monomorphism with cokernel in $\mathcal{C} \cap \mathcal{W}$, and a trivial fibration if f is an epimorphism with kernel in $\mathcal{F} \cap \mathcal{W}$. But weak equivalences do not have to be monos or epis, so we can't define them in the same way. Instead, we define f to be a weak equivalence if it is the composition of a trivial cofibration followed by a trivial fibration.

There are now a great many things to check. Just to give the flavor of the argument, we prove a few results needed for Theorem 2.5.

Lemma 2.6. *Cofibrations, trivial cofibrations, fibrations, and trivial fibrations are all closed under compositions.*

Proof. Suppose i and j are cofibrations. We have a short exact sequence

$$0 \to \operatorname{cok} i \to \operatorname{cok} ij \to \operatorname{cok} j \to 0.$$

(special case of the snake lemma). Because $\mathcal{C}$ is the left half of a cotorsion theory, it is closed under extensions. Thus $\operatorname{cok} ij \in \mathcal{C}$ and so

ij is a cofibration. Because $\mathcal{W}$ is thick, if i and j are trivial cofibrations, so is ij. The fibration case is similar. $\square$

Proposition 2.7. *Every map f can be factored as $f = qj = pi$, where j is a cofibration, q is a trivial fibration, i is a trivial cofibration, and p is a fibration.*

Proof. This proceeds in stages. The two cases are similar, so we just do the qj case. We first assume $f\colon A \to B$ is a monomorphism already, with cokernel C. Since $(\mathcal{C}, \mathcal{F}\cap\mathcal{W})$ is a complete cotorsion theory, there is a surjection $QC \to C$ where $QC \in \mathcal{C}$, with kernel K in $\mathcal{F} \cap \mathcal{W}$. By taking the pullback, we get a monomorphism $j\colon A \to B'$ with cokernel QC, so j is a cofibration. We also get $q\colon B' \to B$, which is a surjection with kernel K, so a trivial fibration as required.

Now suppose f is an epimorphism with kernel K. Then we can repeat the same trick, using an embedding $K \to RK$ with $RK \in \mathcal{F} \cap \mathcal{W}$ and cokernel $C \in \mathcal{C}$, and taking the pushout instead of the pullback.

Now, for an arbitrary map f, we write it as the composite

$$A \xrightarrow{i_1} A \oplus B \xrightarrow{f+1_B} B$$

of a monomorphism followed by an epimorphism. Write $f + 1_B = q'j'$, where q' is a trivial fibration and j' is a cofibration. Then write $j'i_1 = q''j$, where q'' is a trivial fibration and j is a cofibration. Take $q = q''q'$ to complete the proof. $\square$

Proposition 2.8. *Weak equivalences as defined above are closed under compositions.*

Proof. It suffices to check that a composition of the form ip, where p is a trivial fibration and i is a trivial cofibration, can be written $ip = qj$, where q is a trivial fibration and j is a trivial cofibration. By the preceding proposition, we can write $ip = qj$, where q is a trivial fibration and j is a cofibration. This gives us the diagram below

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "j"] \arrow[d, "p"'] & W \arrow[r] \arrow[d, "q"'] & \operatorname{cok} j \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"'] & Z \arrow[r] & \operatorname{cok} i \arrow[r] & 0
\end{tikzcd}$$

which leads to the short exact sequence

$$0 \to \ker p \to \ker q \to \ker r \to 0.$$

Since p and q are trivial fibrations, $\ker p$ and $\ker q$ are in $\mathcal{W}$. Since $\mathcal{W}$ is thick, $\ker r \in \mathcal{W}$. But $\operatorname{cok} i \in \mathcal{W}$ since i is a trivial cofibration.

We conclude that $\operatorname{cok} j \in \mathcal{W}$ since $\mathcal{W}$ is thick, and hence j is a trivial cofibration as required. $\square$

3. Cofibrant generation

So now we have this correspondence between abelian model categories and compatible pairs of complete cotorsion theories. We should then ask: given an important property of model categories, how is that property reflected in the compatible pair of complete cotorsion theories?

For example, when is our abelian model structure cofibrantly generated? Recall that a model structure is **cofibrantly generated** when there is a set I of cofibrations and a set J of trivial cofibrations such that p is a trivial fibration if and only if it has the right lifting property with respect to I, and p is a fibration if and only if it has the right lifting property with respect to J. (Plus an additional smallness condition that we omit, because it is automatically satisfied in any standard algebraic category; it is only topologies that make this one hard). The key thing here is that we do not need the entire proper class of cofibrations to detect the trivial fibrations, but just the **set** I.

The translation between the abelian model structures and cotorsion theories basically takes a cofibration to its cokernel, so we define a cotorsion theory $(\mathcal{D}, \mathcal{E})$ to be **cogenerated by a set** when there is a subset $\mathcal{D}'$ of the class $\mathcal{D}$ such that $E \in \mathcal{E}$ if and only if $\operatorname{Ext}^1(D, E) = 0$ for all $D \in \mathcal{D}'$. This definition was actually made by Eklof and Trlifaj [ET01] without knowing anything about model categories.

For example, the (projective, everything) cotorsion theory is cogenerated by 0 in any abelian category, and the (everything, injective) cotorsion theory is cogenerated by the set of all $R/\mathfrak{a}$ in the category of R-modules. (This is Baer's criterion for injectivity).

Then the following lemma is not difficult.

Lemma 3.1. *If an abelian model category is cofibrantly generated, then the corresponding complete cotorsion theories* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *and* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *are each cogenerated by a set.*

We would like the converse to be true as well. In fact, we want more than that. Recall that the point of a model category being cofibrantly generated is then Quillen's small object argument gives you the factorization axioms for free. So we want to start with two compatible cotorsion theories, not necessarily complete, but cogenerated by a set, and argue that the cotorsion theories are automatically complete, and hence we get an abelian model structure. In fact, this seems to be true in practice, but the simplest theorem along these lines requires a strong hypothesis.

Proposition 3.2. *If $\mathcal{A}$ is a Grothendieck category with enough projectives, then every cotorsion theory cogenerated by a set is complete. Furthermore, given a pair of compatible cotorsion theories each cogenerated by a set, the corresponding abelian model structure is cofibrantly generated.*

I think of the Grothendieck hypothesis as the best hypothesis on an abelian category. It is general enough to include categories that occur frequently in algebraic topology (sheaves and comodules, for example), but strong enough to ensure good properties. An abelian category is Grothendieck when it has a generator and filtered colimits are exact.

The reason for having projectives is so that, given one of your cogenerators C, you have a good choice for a monomorphism whose cokernel is C. Usually, even when you do not have enough projectives, you actually do have such a good choice anyway, but it is more complicated to make this into a theorem.

4. Monoidal structure

One of the most important properties a model structure can have is compatibility with a tensor product. This is particularly important in the algebraic situation. For example, given any Grothendieck category $\mathcal{A}$, there is an injective model structure on unbounded chain complexes over $\mathcal{A}$. The cofibrations are the monomorphisms, the weak equivalences are the homology isomorphisms, and the fibrations are the epimorphisms with DG-injective kernel. (DG-injective means each entriy is injective, and every map from an exact complex into it is chain homotopic to 0). This is the foundation for homological algebra of the Ext sort in any Grothendieck category.

But as a practical matter, one almost always has a tensor product around; the tensor product of modules, or sheaves, or comodules. And injective resolutions are almost never compatible with the tensor product. So if you want to work with Tor or some kind of derived tensor product, you cannot use the injective model structure.

In general, we have the following definition.

Definition 4.1. A model structure on a symmetric monoidal category $\mathcal{A}$ is called **monoidal** whenever the following conditions hold:

(1) Given cofibrations $i\colon A \to B$ and $j\colon C \to D$, the induced map

$$i \,\square\, j\colon (A \otimes D) \amalg_{A \otimes C} (B \otimes C) \to B \otimes D$$

is a cofibration, which, in addition, is a trivial cofibration if either i or j is trivial.
(2) An annoying condition that only arises when the unit of the tensor product is not cofibrant.

The definition of a monoidal model category was not really even forumlated precisely until the late 1990's, although it is based on Quillen's definition of a simplicial model category dating to the 1960's. Looking back on it, however, one can say that one of the biggest problems in algebraic topology was the failure to find a monoidal model category whose homotopy category is the usual stable homotopy category. This problem was solved in the 1990's by Elmendorf, Kriz, Mandell, and May [EKMM97] and Smith [HSS00].

The projective model structure on chain complexes of R-modules is monoidal, but, as mentioned above, the injective model structure is not.

Here is what a monoidal abelian model structure looks like from the cotorsion theory point of view.

Theorem 4.2. *Let $\mathcal{A}$ be an abelian model category,and suppose $\mathcal{A}$ is closed symmetric monoidal. Suppose the following conditions are satisfied*:

(1) *Every element of $\mathcal{C}$ is flat.*
(2) *If $X, Y \in \mathcal{C}$, then $X \otimes Y \in \mathcal{C}$.*
(3) *If $X, Y \in \mathcal{C}$ and one of them is in $\mathcal{W}$, then $X \otimes Y \in \mathcal{W}$.*
(4) *The unit S is in $\mathcal{C}$.*

Then $\mathcal{A}$ is a monoidal model category.

Here "flat" means what it usually does. That is, X is flat if the functor $X \otimes (-)$, which is right exact since it is a left adjoint, is actually exact.

5. Standard examples

Having done the work of relating abelian model categories to pairs of complete cotorsion theories, we now consider the standard examples of model structures on abelian categories.

Perhaps the simplest example of a model category is the category of R-modules when R is a Frobenius ring. This means that projective and injective modules coincide. The standard example is the group ring $R = k[G]$ of a finite group G over a field k. In this case, we can take $\mathcal{C} = \mathcal{F}$ to be the entire category of R-modules, and take $\mathcal{W}$ to be the class of projective (=injective) modules, which is thick in this unusual case. The two complete cotorsion theories are then (everything, projective=injective) and (projective=injective, everything). The homotopy category of this model category is called the **stable category of R-modules** and is the main object of study in modular representation theory (Benson, Carlson, Rickard). Note that it is mildly painful to say exactly what a stable equivalence is, just as it is difficult to say what a weak equivalence is in the above correspondence between pairs of complete cotorsion theories and abelian model categories.

The first model structure ever constructed was Quillen's model structure on the category $\mathrm{Ch}(R)_+$ of nonnegatively graded chain complexes of R-modules. From our point of view, this model structure corresponds to $\mathcal{C}$ being the dimensionwise projective complexes, $\mathcal{W}$ being the exact complexes, and $\mathcal{F}$ being everything. The complete cotorsion theory $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is then (dimensiowise projective, exact) and the complete cotorsion theory $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is (projective, everything). The work of the proof is showing the first one is a complete cotorsion theory and that dimensionwise projective and exact implies projective (which, for a chain complex, implies contractible). There is an analogous model structure, also due to Quillen, on nonnegatively graded chain complexes where $\mathcal{C}$ is everything and $\mathcal{F}$ is dimensionwise injective complexes.

Now suppose $\mathcal{A}$ is a Grothendieck category. As mentioned above, there is an injective model structure on $\mathrm{Ch}(\mathcal{A})$, the category of unbounded chain complexes on $\mathcal{A}$. Here $\mathcal{C}$ is everything, $\mathcal{W}$ is the exact complexes, and $\mathcal{F}$ consists of the DG-injective complexes (dimensionwise injective complexes such that every map from an exact complex in is chain homotopic to 0). Again, a complex is DG-injective and exact if and only if it is actually injective, so the cotorsion theory $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is (everything, injective). The cotorsion theory $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is (exact, DG-injective).

The dual thing works for $\mathrm{Ch}(R)$, for R a ring. That is, we take $\mathcal{C}$ to be DG-projectives (dimensionwise projectives such that every map to an exact is chain homotopic to 0), $\mathcal{F}$ to be everything, and $\mathcal{W}$ to be exact complexes. Again, something that is both DG-projective and exact is actually projective.

6. Gorenstein rings

Here is a new example of an abelian model category from [Hov02]. The idea here is that we would like to do modular representation theory over the integers instead of over fields. So we want to study $\mathbb{Z}[G]$, when G is a finite group. This is no longer a Frobenius ring; projectives and injectives do not coincide. However, it does have some exceptionally nice properties: it is left and right Noetherian, and $\mathbb{Z}[G]$, while not self-injective, does have finite injective dimension as either a left or right module over itself. (This was first noticed by Eilenberg and Nakayama [EN55]). Such a ring is called a **Gorenstein ring**, or an **Iwanaga-Gorenstein ring**. It is a reasonable generalization of the usual notion of a commutative Gorenstein ring.

The salient fact about Gorenstein rings is that in a Gorenstein ring, the modules of finite projective dimension and the modules of finite injective dimension coincide (and the maximum injective or projective dimension is the injective dimension of R). This is due to Iwanaga [Iwa79]. It is easy to prove from this that these modules form a thick subcategory. We then define $\mathcal{W}$ to be this class of finite projective dimension modules, in analogy to the Frobenius case.

But now the analogy breaks down a little, as we cannot expect to get a model structure in which every module is both cofibrant and fibrant. If we want every module to be fibrant, then we take $\mathcal{C}$ to be the class of **Gorenstein projective** modules; these are, of course, modules P for which $\operatorname{Ext}^1(P, W) = 0$ for all W of finite projective dimension. Let d be the injective dimension of R. Then a typical Gorenstein projective is a dth syzygy of an arbitrary module. That is, if we take a module M and take a partial projective resolution

$$0 \to K \to P_{d-1} \to \cdots \to P_0 \to M \to 0$$

where the P_i are projective, then K is Gorenstein projective. These modules have been studied before; when they are finitely generated, they are called **maximal Cohen-Macaulay modules**.

There is a dual notion of a **Gorenstein injective** module. Here I is Gorenstein injective if and only if $\operatorname{Ext}^1(W, I) = 0$ for all W of finite projective dimension. A typical Gorenstein injective module is a dth cosyzygy of an arbitrary module.

We then get, after some work of course, two (Quillen equivalent) model structures on the category of R-modules when R is Gorenstein. Both model categories have the same class of trivial objects $\mathcal{W}$, the modules of finite projective dimension. In the projective model structure, everything is fibrant, and M is cofibrant if and only if it is Gorenstein projective. In the injective model structure, everything is cofibrant, and M is fibrant if and only if it is Gorenstein injective.

The resulting homotopy category has every right to be called the **stable category of R-modules**. It is a triangulated category, and when $R = K[G]$ and K is a principal ideal domain, it has a good closed symmetric monoidal structure (given by tensoring over K).

As far as I know, not very much is known about this stable module category. There are many results about the stable module category of $k[G]$ when k is a field, such as a classification of the thick subcategories of small objects (=finitely generated modules) [BCR97]. It would be good to know how much different the classification over $K[G]$ is.

7. Gillespie's work

The results in this section are due to my student, Jim Gillespie, and come from [Gil04b], [Gil04a], and personal communications.

7.1. **The general approach.** Gillespie looks at the general question of the relationship between a cotorsion theory on a Grothendieck category $\mathcal{A}$ and the homological algebra of $\mathcal{A}$. That is, given a single cotorsion theory $(\mathcal{D}, \mathcal{E})$, can we induce a model structure on $\mathrm{Ch}(\mathcal{A})$ from this cotorsion theory on $\mathcal{A}$? We know two cases of this already: the (projective, everything) cotorsion theory on $\mathcal{A}$ corresponds to the projective model structure on $\mathrm{Ch}(\mathcal{A})$, when it exists, and the (everything, injective) cotorsion theory on $\mathcal{A}$ corresponds to the injective model structure on $\mathrm{Ch}(\mathcal{A})$.

Recall how this works for the (projective, everything) model structure. The two cotorsion theories on $\mathrm{Ch}(\mathcal{A})$ in this case are (projective, everything) and (DG-projective, exact). There is of course a categorical definition of projective in $\mathrm{Ch}(\mathcal{A})$, but that will be of no help for a more general cotorsion theory. Instead, note that a complex X is projective if and only if X is exact and Z_nX is projective for all n. This suggests the following definition.

Definition 7.1. Suppose $\mathcal{D}$ is a class of objects in a bicomplete abelian category $\mathcal{A}$. Define $\widetilde{\mathcal{D}}$ to be the class of objects X in $\mathrm{Ch}(\mathcal{A})$ such that X is exact and $Z_nX \in \mathcal{D}$ for all n.

So if $\mathcal{D}$ is projectives, we recover the notion of a projective complex. If $\mathcal{D}$ is everything, we recover the notion of an exact complex.

We still have to recover the notion of DG-projective. Recall that X is DG-projective if each X_n is projective and any map from X to an exact complex is chain homotopic to 0. This suggests the following definition.

Definition 7.2. Suppoe $(\mathcal{D}, \mathcal{E})$ is a cotorsion theory in a bicomplete abelian category $\mathcal{A}$. Define dg-$\widetilde{\mathcal{D}}$ to be the class of all X in $\mathrm{Ch}(\mathcal{A})$ such that $X_n \in \mathcal{D}$ for all n and every map from X to a complex in $\widetilde{\mathcal{E}}$ is chain homotopic to 0. Similarly, define dg-$\widetilde{\mathcal{E}}$ to be the class of all

$X \in \mathrm{Ch}(\mathcal{A})$ such that $X_n \in \mathcal{E}$ for all n and every map from a complex in $\widetilde{\mathcal{D}}$ to X is chain homotopic to 0.

So if $(\mathcal{D}, \mathcal{E})$ is (projectives, everything), then dg-$\widetilde{\mathcal{D}}$ is the class of DG-projectives and dg-$\widetilde{\mathcal{E}}$ is everything. Similarly, if $(\mathcal{D}, \mathcal{E})$ is (everything, injectives), then $\widetilde{\mathcal{D}}$ is the class of exact complexes, $\widetilde{\mathcal{E}}$ is the class of injective complexes, dg-$\widetilde{\mathcal{D}}$ is everything, and dg-$\widetilde{\mathcal{E}}$ is the class of DG-injective complexes.

Now, the goal of Gillespie's work is to prove a metatheorem of the following sort:

Theorem 7.3. *If $(\mathcal{D}, \mathcal{E})$ is a nice enough cotorsion theory on a bicomplete abelian category $\mathcal{A}$, then there is an induced abelian model structure on $Ch(\mathcal{A})$, where $\mathcal{C} = \text{dg-}\widetilde{\mathcal{D}}$, $\mathcal{F} = \text{dg-}\widetilde{\mathcal{E}}$, and $\mathcal{W}$ is the class of exact complexes.*

Of course, he also wants to give nontrivial examples of this theorem.

7.2. **Making the theorem concrete.** We now need to specify precisely what it means for a cotorsion theory $(\mathcal{D}, \mathcal{E})$ to be "nice enough" in Theorem 7.3.

The first thing to verify is that $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are indeed cotorsion theories. This is simple enough that we can do it here, for $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$.

We first show that $\mathrm{Ext}^1(Y, X) = 0$ for $Y \in \widetilde{\mathcal{D}}$ and $X \in \text{dg-}\widetilde{\mathcal{E}}$. So suppose we have a short exact sequence of complexes

$$0 \to X \to W \to Y \to 0$$

with $X \in \text{dg-}\widetilde{\mathcal{E}}$ and $Y \in \widetilde{\mathcal{D}}$. Then each X_n is in $\mathcal{E}$ and each Y_n is in $\mathcal{D}$ (because $Z_n Y \in \mathcal{D}$ for all n and Y is exact, so Y_n is an extension of $Z_n Y$ and $Z_{n-1} Y$). Therefore, our short exact sequence of complexes is dimensionwise split, so $W_n \cong X_n \oplus Y_n$. In terms of this decomposition, the differential on W is $d = (d_X, \tau + d_Y)$, where $\tau \colon Y_n \to X_{n-1}$. Because $d^2 = 0$, we see that $\tau \colon Y \to \Sigma X$ is a chain map. By hypothesis, this chain map is chain homotopic to 0. The chain homotopy can then be used to define a splitting of our sequence by a chain map.

Now suppose $\mathrm{Ext}^1(Y, X) = 0$ for all $X \in \text{dg-}\widetilde{\mathcal{E}}$. We want to show that $Y \in \widetilde{\mathcal{D}}$. The first thing to point out is that

$$\mathrm{Ext}^1(Y, D^{n+1} A) \cong \mathrm{Ext}^1(Y_n, A).$$

(Just draw what an extension of complexes looks like). It follows easily from this that $Y_n \in \mathcal{D}$ for all n, since $D^n A \in \text{dg-}\widetilde{\mathcal{E}}$ whenever $A \in \mathcal{E}$.

Given this, an element of $\operatorname{Ext}^1(Y, S^{n-1}A)$ is determined by a map $Y_n/B_nY \to A$ (this is the same as a chain map $Y \to S^nA$). However, two maps determine the same extension if they are chain homotopic as chain maps $Y \to S^nA$. Said another way, $\operatorname{Ext}^1(Y, S^{n-1}A)$ is the quotient $\operatorname{Hom}(Y_n/B_nY, A)/\operatorname{Hom}(Y_{n-1}, A)$. If this quotient is to be 0 for all $A \in \mathcal{E}$, we can in particular take A to be an injective envelope of Y_n/B_nY to see that Y is exact. But then $\operatorname{Ext}^1(Y, S^{n-1}A)$ is isomorphic to $\operatorname{Ext}^1(Z_{n-1}Y, A)$, from which we see that $Z_{n-1}Y \in \mathcal{D}$ for all n. Thus $Y \in \widetilde{\mathcal{D}}$ as required.

A similar, but simpler, argument shows that if $\operatorname{Ext}^1(Y, X) = 0$ for all $Y \in \widetilde{\mathcal{D}}$, then $X \in \text{dg-}\widetilde{\mathcal{E}}$. For this, one uses the isomorphism

$$\operatorname{Ext}^1(D^nA, X) \cong \operatorname{Ext}^1(A, X_n)$$

to see that $X_n \in \mathcal{E}$ for all n. It then follows that any element in $\operatorname{Ext}^1(Y, X)$ is dimensionwise split for $Y \in \widetilde{\mathcal{D}}$, so $\operatorname{Ext}^1(Y, X)$ is isomorphic to chain homotopy classes of chain maps from Y to ΣX. Since $\operatorname{Ext}^1(Y, X) = 0$, we see that $X \in \text{dg-}\widetilde{\mathcal{E}}$.

Now, if we worked with $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ instead, we would have run into a problem. In the above argument, there was a point where we embedded Y_n/B_nY into an element of $\mathcal{E}$, which we can do by taking an injective. The dual will cause us trouble because we do not want to assume there are enough projectives in $\mathcal{A}$. So instead we assume there are enough $\mathcal{D}$-objects in $\mathcal{A}$, in the sense that everything in $\mathcal{A}$ is a quotient of something in $\mathcal{D}$. This would be automatic if $(\mathcal{D}, \mathcal{E})$ were a complete cotorsion theory.

So we get the following proposition of Gillespie.

Proposition 7.4. *If $(\mathcal{D}, \mathcal{E})$ is a cotorsion theory on an abelian category $\mathcal{A}$ that has enough $\mathcal{D}$-objects, then $(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$ and $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ are cotorsion theories on $Ch(\mathcal{A})$.*

We now want to know whether these cotorsion theories are compatible with the class $\mathcal{W}$ of exact complexes. That is, we want to know that

$$\text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W} = \widetilde{\mathcal{D}} \text{ and } \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W} = \widetilde{\mathcal{E}}.$$

It is fairly straightforward to show the inclusions

$$\widetilde{\mathcal{D}} \subseteq \text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W} \text{ and } \widetilde{\mathcal{E}} \subseteq \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W}.$$

One shows that any map from something in $\widetilde{\mathcal{D}}$ to something in $\widetilde{\mathcal{E}}$ is chain homotopic to 0. The idea for the converse is as follows. Given $X \in \text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W}$, we want to show $\operatorname{Ext}^1(Z_nX, A) = 0$ for all $A \in \mathcal{E}$.

Since we have the short exact sequence

$$0 \to Z_{n+1}X \to X_{n+1} \to Z_n X \to 0$$

amd $X_{n+1} \in \mathcal{D}$, it suffices to show that any map $Z_{n+1}X \to A$ extends to a map $X_{n+1} \to A$. Take an augmented injective resolution I_* of A (so $I_0 = A$ and I_{-1} is an injective envelope of A). With any justice, this should be a complex in $\widetilde{\mathcal{E}}$, since A was in $\mathcal{E}$ to start with. Then a map $Z_{n+1}X \to A$ induces a map of complexes $\Sigma^{-n-2}X \to I_*$ using injectivity. This chain map is chain homotopic to 0, and the chain homotopy gives us an extension $X_{n+1} \to A$.

This argument depended on I_* actually being in $\widetilde{\mathcal{E}}$. This is **NOT** automatic, however. We call a cotorsion theory **hereditary** if any of the following three equivalent conditions are satisfied.

(1) $\operatorname{Ext}^i(D, E) = 0$ for all $D \in \mathcal{D}$, $E \in \mathcal{E}$, and $i > 0$.
(2) $\mathcal{D}$ is closed under kernels of epimorphisms.
(3) $\mathcal{E}$ is closed under cokernels of monomorphisms.

Every cotorsion theory that arises naturally is hereditary; certainly the flat cotorsion theory is, for example.

Then we have the following proposition, again due to Gillespie.

Proposition 7.5. *Suppose $(\mathcal{D}, \mathcal{E})$ is a hereditary cotorsion theory in a Grothendieck category $\mathcal{A}$ with enough $\mathcal{D}$-objects. Then* dg-$\widetilde{\mathcal{D}} \cap \mathcal{W} = \widetilde{\mathcal{D}}$. *If, in addition, $\mathcal{A}$ has enough projectives, then* dg-$\widetilde{\mathcal{E}} \cap \mathcal{W} = \widetilde{\mathcal{E}}$.

As a practical matter, though, most of the categories we are interested in do not have enough projectives. Gillespie and Ed Enochs get around this with a transfinite induction argument that gives the following proposition.

Proposition 7.6. *Suppose $(\mathcal{D}, \mathcal{E})$ is a cotorsion theory that is cogenerated by a set on a Grothendieck category $\mathcal{A}$ with enough $\mathcal{D}$-objects. Then* (dg-$\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}}$) *has enough injectives.*

One can look on this proposition as a variant of the small object argument, but it is much more complicated to prove. Also, it does not seem to work for ($\widetilde{\mathcal{D}}$, dg-$\widetilde{\mathcal{E}}$).

From this, then, we get the following proposition of Gillespie.

Proposition 7.7. *Suppose $(\mathcal{D}, \mathcal{E})$ is a hereditary torsion theory cogenerated by a set on a Grothendieck category $\mathcal{A}$ with enough $\mathcal{D}$-objects. Then*

$$\text{dg-}\widetilde{\mathcal{D}} \cap \mathcal{W} = \widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W} = \widetilde{\mathcal{E}},$$

and (dg-$\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}}$) *is complete.*

The proof is not hard. Suppose $X \in \text{dg-}\widetilde{\mathcal{E}} \cap \mathcal{W}$. We have a short exact sequence

$$0 \to X \to W \to Y \to 0$$

with $W \in \widetilde{\mathcal{E}}$ and $Y \in \text{dg-}\widetilde{\mathcal{D}}$. But then X and W are exact, so Y is too. Thus $Y \in \widetilde{\mathcal{D}}$. This means the sequence splits, so X is a summand in W. But then $X \in \widetilde{\mathcal{E}}$.

Given that $(\text{dg-}\widetilde{\mathcal{D}}, \widetilde{\mathcal{E}})$ has enough injectives, we can use a pushout trick to show it has enough projectives as well, using the fact that there are enough $\mathcal{D}$-objects. That is, you first show that $\text{Ch}(\mathcal{A})$ has enough $\widetilde{\mathcal{D}}$-objects. Then, given X, you take a surjection $A \to X$ with kernel K, where $A \in \widetilde{\mathcal{D}}$. Then you embed K in an element of $\widetilde{\mathcal{E}}$ with cokernel in $\text{dg-}\widetilde{\mathcal{D}}$, and you take the pushout.

So to complete Gillespie's program, we just have to ensure that

$$(\widetilde{\mathcal{D}}, \text{dg-}\widetilde{\mathcal{E}})$$

is complete. In fact, the pushout trick above shows that we only need to be sure it has enough injectives. This appears to be the heart of the matter.

One always wants to use some version of the small object argument of Quillen. But it just seems to be harder than it is for model categories, and being cogenerated by a set does not seem to be enough. So Gillespie, following Enochs and Lopéz-Ramos [ELR02], strengthens the definition a bit.

Definition 7.8. A class $\mathcal{D}$ is a called a **Kaplansky class** if there is some cardinal κ such that, for every κ-generated subobject T of an object $D \in \mathcal{D}$, there is a κ-presentable object $S \in \mathcal{D}$ such that $T \subseteq S \subseteq D$ and $D/S \in \mathcal{D}$.

In an arbitrary category, an object A is κ-generated if $\text{Hom}(A, -)$ commutes with λ-fold coproducts for all κ-filtered ordinals λ (any regular cardinal larger than κ is κ-filtered). On the other hand, A is κ-presentable if $\text{Hom}(A, -)$ commutes with all κ-filtered colimits. The easiest case is when $\kappa = \omega$, when we do recover the usual definition of finitely generated and finitely presentable, only without reference to a specific generator of the category.

This is a strange definition at first. It is motivated by flat modules, where it asserts that, given any small subset of a flat module, there is a flat submodule that contains it and sits inside the big module purely. This was the key idea in the proof of the flat cover conjecture by Bican, El Bashir, and Enochs [BEBE01].

We can now state the precise version of Gillespie's main theorem.

Theorem 7.9. *Suppose $(\mathcal{D}, \mathcal{E})$ is a hereditary cotorsion theory such that $\mathcal{D}$ is a Kaplansky class on a Grothendieck category $\mathcal{A}$ with enough $\mathcal{D}$-objects. Then there is an induced abelian model structure on Ch$(\mathcal{A})$, where $\mathcal{C} = \text{dg-}\widetilde{\mathcal{D}}$, $\mathcal{F} = \text{dg-}\widetilde{\mathcal{E}}$, and $\mathcal{W}$ is the class of exact complexes.*

7.3. **Sheaves and schemes.** The motivation and application of Gillespie's work was the category of sheaves on a ringed space and the category of quasi-coherent sheaves on a scheme. Recall that a ringed space is a topological space S equipped with a sheaf of rings $\mathcal{O}$; that is, a contravariant functor from open sets of S to commutative rings that is locally determined (the sheaf property). A one-point ringed space is of course a commutative ring. The category of $\mathcal{O}$-modules is therefore a generalization of the category of R-modules for a commutative ring R; here an $\mathcal{O}$-module M is a sheaf of abelian groups over S such that $M(U)$ is naturally a module over $\mathcal{O}(U)$. The category of $\mathcal{O}$-modules has a lot in common with the category of R-modules; it is a closed symmetric monoidal Grothendieck category.. The tensor product you can think of as being defined stalkwise in the obvious way.

Therefore, we would expect $\mathrm{Ch}(\mathcal{O})$ to be a symmetric monoidal model category, so that the derived category of $\mathcal{O}$-modules inherits a tensor product. However, before Gillespie's work, I don't believe this was known. The injective model structure certainly exists on $\mathrm{Ch}(\mathcal{O})$, but it is not compatible with the tensor product, and cannot be used to define a derived tensor product. The projective model structure only exists rarely, because generally there are not enough projective $\mathcal{O}$-modules. There are enough flats, though; in fact, the flat sheaves $\mathcal{O}_U$ generate the category, where the stalks of $\mathcal{O}_U$ agree with the stalks of $\mathcal{O}$ inside U and are 0 outside U. The author used these sheaves to construct a monoidal model structure on $\mathrm{Ch}(\mathcal{O})$ in [Hov01], but only under an annoying technical assumption on the ringed space, involving the finiteness of sheaf cohomology.

Gillespie's work allows one to use all the flat sheaves at once, rather than just the ones one can explicitly write down. That is, we start with the (flat, cotorsion) cotorsion theory on $\mathcal{O}$-modules. Using the approach to the flat cover conjecture of [BEBE01], Gillespie shows that that the flat $\mathcal{O}$-modules form a Kaplansky class. (The proof involves purity in an essential way). Hence Theorem 7.9 gives us an abelian model structure on $\mathrm{Ch}(\mathcal{O})$, which Gillespie proves is compatible with the tensor product.

In algebraic geometry, however, it is more common to use the category of quasi-coherent sheaves on a scheme (if your algebraic geometry is rusty, a scheme is a special kind of ringed space built from the prime spectra of commutative rings in the same way that manifolds are built from copies of $\mathbf{R}^n$). This is because if your ringed space is Spec R, then a quasi-coherent sheaf is equivalent to an R-module, whereas an arbitrary sheaf could be more complicated. The word quasi-coherent can be thought as meaning locally a quotient of free sheaves. In this case, Enochs, Estrada, Garcá Rozas, and Oyonarte [EERO00] have proved a result equivalent to the fact that flat quasi-coherent sheaves form a Kaplansky class. There is an additional complication though; it is much less obvious that there are enough flat quasi-coherent sheaves (the sheaves $\mathcal{O}_U$ are not quasi-coherent). This is known by algebraic geometers when the scheme is quasi-compact and separated, and Gillespie and I suspect it holds when the scheme is quasi-compact and quasi-separated (this seems to be the hypothesis of choice in algebraic geometry anyway). But in any case, Gillespie's work then leads to an abelian monoidal model structure of chain complexes of quasi-coherent sheaves over a quasi-compact, separated scheme, and hence a derived tensor product on the derived category. This derived tensor product is used frequently by algebraic geometers, and this provides a simple reason for its existence. (I believe, though I am not an expert, that the usual approach is to patch together the derived tensor products of each affine piece of your scheme).

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