Morita theory for Hopf algebroids

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$\mathbf{Aff}$ = opposite category of commutative rings

Hopf algebroid = pair of commutative rings (A, Γ) such that (A, Γ) corepresents a presheaf $\mathbb{G}$ of groupoids on Aff

$\mathrm{ob}\,\mathbb{G}(R) = \mathbf{Rings}(A, R)$

$\mathrm{mor}\,\mathbb{G}(R) = \mathbf{Rings}(\Gamma, R)$.

The operation "identity morphism" is corepresented by $\epsilon\colon \Gamma \to A$, the counit.

The operation "source" is corepresented by $\eta_L\colon A \to \Gamma$, the left unit.

The operation "target" is corepresented by $\eta_R\colon A \to \Gamma$, the right unit.

The operation "compose" is corepresented by $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$, the comultiplication. Note this is a bimodule tensor product.

The operation "inverse" is corepresented by $\chi\colon \Gamma \to \Gamma$, the conjugation.

If E is a ring spectrum such that E_*E is flat over E_*, then (E_*, E_*E) is a Hopf algebroid.

The counit is induced by the multiplication $E \wedge E \to E$.

The left unit is induced by $E = E \wedge S \to E \wedge E$.

The right unit is induced by $E = S \wedge E \to E \wedge E$.

The comultiplication is induced by $E \wedge E = E \wedge S \wedge E \to E \wedge E \wedge E$ and the isomorphism $E_*(E \wedge E) \cong E_*E \otimes_{E_*} E_*E$.

The conjugation is induced by the twist $E \wedge E \to E \wedge E$.

This works for $E = MU$ and theories derived from it such as $BP, E(n), K, K(n), H\mathbb{Z}/p$, but not $KO, EO(2)$.

A *comodule* over a Hopf algebroid (A, Γ) is an A-module M equipped with a coassociative, counital coaction map $\psi\colon M \to \Gamma \otimes_A M$.

If E is a ring spectrum such that E_*E is flat over E_*, then E_*X is a comodule over (E_*, E_*E) for any spectrum X.

The coaction is induced by $E \wedge X = E \wedge S \wedge X \to E \wedge E \wedge X$ and the isomorphism $E_*(E \wedge X) \cong E_*E \otimes_{E_*} E_*X$.

Furthermore, the E_2 term of the Adams spectral sequence based on E is $\mathrm{Ext}_{E_*E}(E_*, E_*Y)$, and this is comodule Ext.

When Γ is flat over A, the category of Γ-comodules is a bicomplete closed symmetric monoidal AB5 category with enough injectives, but is not well understood.

The symmetric monoidal structure is tensor over A, with diagonal coaction, written $M \wedge N$. The unit is A.

The closed structure, like all right adjoints in this category, is difficult to construct.

General method for constructing right adjoints:

The forgetful functor from Γ-comodules to A-modules has a right adjoint that takes M to $\Gamma \otimes M$, the "cofree comodule" on M.

For a comodule M, the coaction $\psi \colon M \to \Gamma \otimes M$ embeds M into a cofree comodule.

Therefore, any comodule M is naturally the kernel of a map of cofree comodules $\Gamma \otimes M \to \Gamma \otimes N$.

It is usually easy to define your right adjoint on cofree comodules, and only a little harder to define it on maps of cofree comodules.

For example, the product of a family of comodules M_i is defined to be the kernel of an appropriately defined map $\Gamma \otimes \prod M_i \to \Gamma \otimes \prod N_i$.

This map is

$$\Gamma \otimes \prod M_i \to \Gamma \otimes \Gamma \otimes \prod M_i \to \Gamma \otimes \prod (\Gamma \otimes M_i) \\ \to \Gamma \otimes \prod (\Gamma \otimes N_i) \to \Gamma \otimes \prod N_i$$

Products are not exact! Thus there may be higher derived functors of the inverse limit of a sequence. Almost nothing is known about these.

A map of Hopf algebroids $\Phi\colon (A, \Gamma) \to (B, \Sigma)$ corepresents a map Φ^* of presheaves of groupoids.

Φ is a pair of ring maps $\Phi_0\colon A \to B$ and $\Phi_1\colon \Gamma \to \Sigma$ satisfying conditions.

Φ is called an *equivalence* when $\Phi^*(R)$ is an equivalence of categories for all R. (Miller, Morava).

$\Phi^*(R)$ is fully faithful for all R when $\Sigma \cong B \otimes_A \Gamma \otimes_A B$. (Miller?)

In this case, the structure maps of (B, Σ) are determined by Φ_0 and the structure maps of (A, Γ).

$\Phi^*(R)$ is essentially surjective for all R when $A \to B \otimes_A \Gamma$ is a split monomorphism. (Miller?)

Theorem (Miller). If $\Phi\colon (A, \Gamma) \to (B, \Sigma)$ is an equivalence of Hopf algebroids, then Φ induces an equivalence of the category of Γ-comodules with the category of Σ-comodules that takes the Γ-comodule M to $B \otimes_A M$.

Hopf algebroids are presheaves of groupoids; could they be sheaves, with respect to some Grothendieck topology on $\mathbf{Aff}$? (Hopkins, ?).

This would mean the object and morphism functors would have to be sheaves.

For any ring A, $\operatorname{Spec} A = \mathbf{Rings}(A, -)$ is a sheaf on $\mathbf{Aff}$ when we give $\mathbf{Aff}$ the *flat topology*.

In this topology, a cover of a ring R is a finite family S_i of flat extensions of R such that $\prod S_i$ is faithfully flat over R.

The fact that $\mathbf{Rings}(A, -)$ is a sheaf in the flat topology is "faithfully flat descent":

Let $S = \prod S_i$. Then $0 \to R \to S \to S \otimes_R S$ is an equalizer diagram of rings.

Thinking of a Hopf algebroid as a sheaf of groupoids leads to a weaker definition of equivalence.

Define a map $f\colon \mathbb{G} \to \mathbb{H}$ to be *sheaf-theoretically essentially surjective* if:

for any R and any object $y \in \mathbb{H}(R)$ there is a cover $\{S_i\}$ of R, elements x_i in $\mathbb{G}(S_i)$, and morphisms $\alpha_i\colon fx_i \to y_i$ in $\mathbb{H}(S_i)$.

A map f of sheaves of groupoids is called an *internal equivalence* (Joyal-Tierney, Jardine, Hollander) if f is fully faithful and sheaf-theoretically essentially surjective.

Call a map $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ of Hopf algebroids a *weak equivalence* when Φ_* is an internal equivalence in the flat topology.

Theorem (H.): Φ is a weak equivalence if and only if $\Sigma \cong B \otimes_A \Gamma \otimes_A B$ and there is a ring map $B \otimes_A \Gamma \to C$ such that $A \to B \otimes_A \Gamma \to C$ expresses C as a faithfully flat extension of A.

This condition appears in work of Hopkins.

Modifying work of Lazard, following Strickland, then leads to the following theorem:

Theorem (H.): Fix a prime p and an integer $n > 0$. Let (A, Γ) denote the Hopf algebroid $(v_n^{-1}BP_*/I_n, v_n^{-1}BP_*BP/I_n)$. Suppose B is a graded ring equipped with a formal group law whose classifying map factors through A. Then the map $(A, \Gamma) \to (B, B \otimes_A \Gamma \otimes_A B)$ is a weak equivalence of Hopf algebroids.

For example, we can taken $B = v_n^{-1}E(m)_*/I_n$ for $m \geq n$.

Main Theorem (H.): Suppose $\Phi\colon (A,\Gamma) \to (B,\Sigma)$ is a weak equivalence of Hopf algebroids. Then Φ induces an equvalence of categories from (A,Γ)-comodules to (B,Σ)-comodules. This equivalence takes M to $B \otimes_A M$.

Hopkins, Miller may well have known some version of this theorem.

Note that the functor that takes M to $B \otimes_A M$ always has a right adjoint, even if Φ is not a weak equivalence. This right adjoint is constructed as we constructed products.

To prove the theorem, we must figure out what the correct geometric definition of a comodule over a sheaf of groupoids is.

Suppose X is a sheaf of sets on $\mathbf{Aff}$ equipped with a Grothendieck topology.

Demazure-Gabriel: An X-model is a pair (R, x) where R is a ring and $x \in X(R)$. (Strickland calls these *points*).

The opposite category of X-models inherits a Grothendieck topology from $\mathbf{Aff}$.

It has a structure presheaf $\mathcal{O}$(of rings) that assigns the ring R to the point (R, x).

A *(pre)sheaf over* X is a (pre)sheaf of $\mathcal{O}$-modules on the opposite category of X-models.

Thus a presheaf M assigns an R-module M_x to every point (R, x) of X. It is a sheaf when, for any cover $\{S_i\}$ of R, we have an equalizer diagram

$$M \to \prod_i M_{x_i} \to \prod_{ij} M_{x_{ij}}.$$

If $X = \operatorname{Spec} A$, so that $X(R) = \mathbf{Rings}(A, R)$, then every A-module M gives rise to a presheaf $\widetilde{M}$ over X, where $\widetilde{M}_x = R \otimes_x M$ for $x \in X(R)$.

This presheaf is *quasi-coherent*; if $(R, x) \to (S, y)$ is a map of points, then $S \otimes_R \widetilde{M}_x \to \widetilde{M}_y$ is an isomorphism.

Lemma (Demazure-Gabriel): The category of A-modules is equivalent to the category of quasi-coherent presheaves over $\operatorname{Spec} A$.

If $X = (X_0, X_1)$ is a presheaf of groupoids on $\mathbf{Aff}$, then we define a (pre)sheaf over X to be a (pre)sheaf M over X_0 together with a natural isomorphism $\phi\colon \operatorname{dom}^* M \to \operatorname{codom}^* M$ of sheaves over X_1 that satisfies the cocycle condition $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha$ for composable morphisms β, α.

That is, if $\alpha \in X_1(R)$ is a morphism from x to y, ϕ_α is an isomorphism of R-modules $M_x \to M_y$. The cocycle condition says that this isomorphism is compatible with composition in the groupoid $X(R)$.

Theorem (H.) There is an equivalence of categories between comodules over a Hopf algebroid (A, Γ) and quasi-coherent presheaves over $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$.

Any quasi-coherent presheaf is isomorphic to $\widetilde{M}$ for some A-module M. The identity map 1 of Γ is a morphism from η_L to η_R.

Given this, we get a coaction via

$$M \xrightarrow{\eta_L \otimes 1} \Gamma_{\eta_L} \otimes_A M \xrightarrow{\phi_1} \Gamma_{\eta_R} \otimes_A M$$

The cocycle condition implies that this action is counital and coassociative.

Conversely, if M is a comodule, and $\alpha\colon \Gamma \to R$ is a morphism from x to y, define ϕ_α to be

$$R_x \otimes M \to R_x \otimes \Gamma \otimes M \to R_x \otimes R_y \otimes M \to R_y \otimes M.$$

Theorem (H.) An internal equivalence $f\colon \mathbb{G} \to \mathbb{H}$ of sheaves of groupoids induces an equivalence from the category of sheaves over $\mathbb{H}$ to the category of sheaves over $\mathbb{G}$.

The equivalence is pullback f^*, where $(f^*M)_x = M_{fx}$.

Proof is essentially a very complicated diagram chase. E.g. for fullness, if $\tau\colon f^*M \to f^*N$ is map of sheaves, we first define $\sigma_y\colon M_y \to N_y$ to be τ_x if $y = fx$, making random choices. Then we extend to y in the essential image of f, then we use covers to extend to y in sheaf-theoretic essential image.

A quasi-coherent sheaf is a sheaf in the flat topology, by faithfully flat descent.

Theorem (H.) An internal equivalence $f\colon \mathbb{G} \to \mathbb{H}$ of sheaves of groupoids in the flat topology induces an equivalence from the category of quasi-coherent sheaves over $\mathbb{H}$ to the category of quasi-coherent sheaves over $\mathbb{G}$.

Full and faithful is easy. For essentially surjective, given quasi-coherent sheaf N, there is a flat sheaf M that hits it up to isomorphism. Show M has to be quasi-coherent.

This is tricky; uses idea of a pure equalizer diagram.

This completes the proof of the Main Theorem.

Applications

The category of $v_n^{-1}E(m)_*E(m)/I_n$-comodules is equivalent to the category of $v_n^{-1}BP_*BP/I_n$-comodules for $m > n$.

This recovers H.-Sadofsky result that $E(n)$-chromatic spectral sequence is first $n+1$ columns of usual chromatic spectral sequence.

New conceptual proof and extension of Miller-Ravenel change of rings theorem:

If M and N are BP_*BP-comodules, v_n acts isomorphically on N, and either M is finitely presented or $N = v_n^{-1}N'$ where N' is finitely presented and I_n-nilpotent, then

$$\operatorname{Ext}_{BP_*BP}(M, N) \cong \operatorname{Ext}_{E(m)_*E(m)}(E(m)_* \otimes_{BP_*} M, E(m)_* \otimes_{BP_*} N)$$

for all $m \geq n > 0$.

Goerss-Hopkins introduced notion of Adams Hopf algebroid. All the Hopf algebroids used in topology are Adams Hopf algebroids.

Theorem (H.) If (A, Γ) is an Adams Hopf algebroid, there is a very nice symmetric monoidal model structure on unbounded complexes of Γ-comodules.

The homotopy category of this model structure is a stable homotopy category; the homotopy of the sphere is $\text{Ext}_{\Gamma}(A, A)$. (H.-Strickland had constructed this stable homotopy category (unpublished)).

A weak equivalence of Hopf algebroids induces a Quillen equivalence of the associated model categories.

Speculation: the final statement of the Miller-Ravenel (Morava?) change of rings theorem will be a statement about an equivalence of stable homotopy categories, involving localization in the BP_*BP stable homotopy category.

(A, Γ) Adams Hopf algebroid means Γ-comodule Γ is a filtered colimit of comodules Γ_i where each Γ_i is finitely generated and projective over A.

Let $\mathcal{G}$ denote the set of Γ-comodules that are finitely generated projective over A. Then (A, Γ) Adams implies that $\mathcal{G}$ is a set of generators for Γ-comodules.

Can build the *projective model structure*. Generating cofibrations are $S^{n-1}P \to D^n P$ for $P \in \mathcal{G}$. Generating trivial cofibrations are $0 \to D^n P$ for $P \in \mathcal{G}$. (Goerss-Hopkins).

Projective equivalences are maps such that Γ-comod(P, f) is a homology iso for all $P \in \mathcal{G}$.

Every projective equivalence is a homology isomorphism but converse is false.

Projective fibrations are maps such that Γ-comod(P, f) is surjective for all $P \in \mathcal{G}$.

Projective cofibrations are degreewise split monomorphisms whose cokernels are cofibrant. X is cofibrant if and only if $X = \bigcup X^n$, where $X^n \to X^{n+1}$ is a degreewise split monomorphism whose cokernel is a complex of relative projectives with no differential. "Relative projective" means retract of direct sum of P's in $\mathcal{G}$.

In the projective homotopy category, the homotopy of the sphere is just A concentrated in degree 0, since A is cofibrant and fibrant.

We want the homotopy of the sphere to be $\mathrm{Ext}_{\Gamma}(A, A)$.

Let LA be the cobar resolution of A.

So $(LA)_{-n} = \Gamma \otimes \overline{\Gamma}^{\otimes n}$.

Define a map f to be a homotopy isomorphism if $LA \wedge f$ is a projective equivalence.

Can do Bousfield localization on projective model structure to obtain homotopy model structure.

Weak equivalences in homotopy model structure are homotopy isomorphisms.

Homotopy fibrant objects are complexes projectively equivalent to a complex of relative injectives.

Homotopy fibrations are projective fibrations whose kernel is homotopy fibrant.

LA is a fibrant replacement of A, though LA is not cofibrant usually.