The structure of $E(n) \ast E(n)$-comodules
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Data for this talk:

(1) A prime $p$

(2) The $p$-local spectrum $BP$ and its associated Hopf algebroid $(BP_{\ast}, BP_{\ast}BP)$, where $BP_{\ast} \cong \mathbb{Z}_\lp[v_1, v_2, \ldots]$. 

(3) A Landweber exact commutative ring spectrum $E$. This means $E_{\ast}$ is a commutative $BP_{\ast}$-algebra and the sequence $(p, v_1, v_2, \ldots)$ is regular on $E_{\ast}$. In this case we have an associated Hopf algebroid $(E_{\ast}, E_{\ast}E)$, where

$$E_{\ast}E \cong E_{\ast} \otimes_{BP_{\ast}} BP_{\ast}BP \otimes_{BP_{\ast}} E_{\ast}.$$ 

Examples are $E(n)$, $v_n^{-1}BP$, $E_n$, $K$, elliptic cohomology.
If $X$ is a space or a spectrum, $BP_*X$ is a $BP_*BP$-comodule and $E_*X$ is an $E_*E$-comodule. There is a map of Hopf algebroids

$$\Phi: (BP_*, BP_*BP) \to (E_*, E_*E)$$

that induces a functor

$$\Phi_*: BP_*BP\text{-comod} \to E_*E\text{-comod}.$$ defined by $\Phi_*M \cong E_* \otimes_{BP_*} M$. Landweber exactness implies that $\Phi_*$ is exact and

$$E_*X \cong \Phi_*(BP_*X)$$

for any spectrum $X$.

Since $\Phi_*$ commutes with all colimits, it ought to have a right adjoint, though this seems not to have been noticed before.
Lemma: The functor $\Phi_*$ has a right adjoint $\Phi^*$.

Proof: One sees easily that

$$\Phi^*(E_*E \otimes_{E_*} M) \cong B_{P*}B_{P} \otimes_{B_{P*}} M.$$ 

An arbitrary $E_*E$-comodule $M$ fits into an exact sequence of comodules

$$0 \to M \overset{\psi}{\to} E_*E \otimes_{E_*} M \overset{\psi g}{\to} E_*E \otimes_{E_*} N$$

where $g: E_*E \otimes_{E_*} M \to N$ is the cokernel of $\psi$. It is not too difficult to define $\Phi^*$ on arbitrary maps of extended comodules, and then, since $\Phi^*$ will have to be left exact, we must define

$$\Phi^* M \cong \ker \Phi^*(\psi g). \square$$
Now consider the counit of this adjunction

$$\epsilon : \Phi_*\Phi^* M \to M.$$ 

Plug in an extended comodule to find

$$\Phi_*\Phi^*(E_*E \otimes_{E_*} M) \cong \Phi_*(BP_*BP \otimes_{BP_*} M)$$

$$\cong E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} M \cong E_*E \otimes_{E_*} M.$$ 

Thus the counit $\epsilon$ is an isomorphism on extended $E_*E$-comodules. Since it is a natural transformation of left exact functors, and every comodule is a kernel of a map of extended comodules, we get the following theorem.

**Theorem:** The counit $\Phi_*\Phi^* M \to M$ is an isomorphism. In particular, $\Phi^*$ defines an equivalence of categories between $E_*E$-comodules and its image, a full subcategory of $BP_*BP$-comodules.
We now want to know: which subcategory of $BP_*BP$-comodules is it?

For this, we look at the other composite $\Phi^*\Phi_*$. One can easily see that this is a left exact idempotent functor, equipped with a natural transformation $M \to \Phi^*\Phi_*M$.

General nonsense tells us that $\Phi^*\Phi_*$ is a localization functor! We denote it by $LE_*$; it is localization with respect to the class of maps $\mathcal{W} = \{f : \Phi^*\Phi_*f \text{ is an iso}\}$. This means that the map $M \to LE_*M$ is in $\mathcal{W}$ and $LE_*M$ is $\mathcal{W}$-local, which means that

$$BP_*BP\text{-comod}(f, LE_*M)$$

is an isomorphism for all $f \in \mathcal{W}$.

Note that $\mathcal{W} = \{f : \Phi_*f \text{ is an iso}\}$ because $\Phi^*$ is an equivalence onto its image. But $\Phi_*$ is exact, so $\mathcal{W} = \{f : \ker f, \coker f \in \mathcal{A}_{E_*}\}$, where $\mathcal{A}_{E_*} = \ker \Phi_* = \{M : \Phi_*M = 0\}$. 5
We are now reduced to computing $A_{E_*} = \ker \Phi_*$. Because $\Phi_*$ is left exact and commutes with colimits, $A_{E_*}$ is a hereditary torsion theory. That is, it is closed under subobjects, quotient objects, extensions, and arbitrary direct sums (and suspensions, in the graded case).

There are some obvious hereditary torsion theories of $BP_*BP$-comodules. Let $T_n$ be the full subcategory of $v_n$-torsion comodules, so that $T_0$ is the $p$-torsion comodules and $T_{-1}$ is all comodules. Take $T_\infty = \{0\}$. One can easily check that the $T_n$ are hereditary torsion theories, and by well-known results of Johnson-Yosimura,

$$T_{-1} \supseteq T_0 \supseteq \cdots \supseteq T_n \supseteq \cdots.$$ 

Theorem: Suppose $T$ is a hereditary torsion theory of $BP_*BP$-comodules, and suppose $T$ contains a nonzero finitely presented comodule. Then $T = T_n$ for some $-1 \leq n < \infty$. 

Let $I_n = (p, v_1, \ldots, v_{n-1})$ as usual, so $I_0 = 0$.

Theorem: Define height $E$ to be the unique $n$ with $E_*/I_n \neq 0$ and $E_*/I_{n+1} = 0$, or $\infty$ if $E_*/I_n \neq 0$ for all $n$. Then $A_{E_*} = T_{\text{height } E}$.

Theorem: If $E$ and $E'$ are two Landweber exact commutative ring spectra of the same height, then the categories of $E_*E$-comodules and $E'_*E'$-comodules are equivalent.

We thus denote $L_{E(n)_*}$ by $L_n$, in analogy with the topological $L_n$. If $E$ has height $n$, then the category of $E_*E$-comodules is equivalent to the category of $L_n$-local $BP_*BP$-comodules.
Corollary (Miller-Ravenel change of rings theorem): Suppose $M$ and $N$ are $v_n^{-1}BP_*(v_n^{-1}BP)$-comodules. Then

$$\Ext^{**}_{v_n^{-1}BP_*(v_n^{-1}BP)}(M, N) \cong \Ext^{**}_{E(n)_*E(n)}(E(n)_* \otimes_{BP_*} M, E(n)_* \otimes_{BP_*} N).$$

The Ext groups are isomorphic because the categories they are taken in are equivalent!

Can also recover change of rings theorems of Morava and Hovey-Sadofsky.

Corollary: Every nonzero $E_*E$-comodule has a nonzero primitive.

Because every $BP_*BP$-comodule does so, including the local ones.
Corollary: The primitives in $E_*/I_k$ are $\mathbb{F}_p[v_k]$ if $k < \text{height } E$ and $\mathbb{F}_p[v_k, v_k^{-1}]$ if $k = \text{height } E$.

This has to be interpreted correctly if $k = 0$.

Corollary: If $I$ is an invariant radical ideal in $E_*$, then $I = I_n$ for some $n \leq \text{height } E$. Converse is true for $E(n)_*$, $E_n^*$, and $v_n^{-1}BP^*$.

Corollary: (Landweber filtration) Every $E_*E$-comodule $M$ that is finitely presented over $E_*$ admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t$$

by subcomodules so that each quotient $M_i/M_{i-1}$ is isomorphic as a comodule to a suspension of $E_*/I_k$ for some $k \leq \text{height } E$, depending on $i$. 

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The functor $L_n$ on $BP_*BP$-comodules is very interesting itself. Note that it is left exact, but has nontrivial right derived functors $L^i_n$.

Theorem: There is a strongly and conditionally convergent spectral sequence of comodules

$$E^{s,t}_2 = (L^s_n BP_* X)_t \Rightarrow BP_{t-s} L_n X,$$

with $d_r : E^{s,t}_r \to E^{s+r,t+r-1}_r$. Every element in $E^{0,t}_2$ coming from $BP_* X$ is a permanent cycle.

Theorem: (1) $L^s_n M = 0$ if $s > n$.

(2) $L_n BP_* / I_k = BP_* / I_k$ for $k < n$ but $L_n BP_* / I_n = v^{-1}_n BP_* / I_n$.

(3) $L^{n-k}_n (BP_* / I_k) = BP_* / (I_k, I_{n+1}^\infty)$, and all other derived functors are 0.

(4) $L^s_n M$ is the Cech cohomology, in the sense of Greenlees and May, of the $BP_*$-module $M$ relative to $I_{n+1}$.  

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