The structure of $E(n)_*E(n)$-comodules

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Data for this talk:

(1) A prime p

(2) The p-local spectrum BP and its associated Hopf algebroid (BP_*, BP_*BP), where $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

(3) A Landweber exact commutative ring spectrum E. This means E_* is a commutative BP_*-algebra and the sequence $(p, v_1, v_2, \ldots)$ is regular on E_*. In this case we have an associated Hopf algebroid (E_*, E_*E), where

$$E_*E \cong E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_*.$$

Examples are $E(n)$, $v_n^{-1}BP$, E_n, K, elliptic cohomology.

If X is a space or a spectrum, BP_*X is a BP_*BP-comodule and E_*X is an E_*E-comodule. There is a map of Hopf algebroids

$$\Phi\colon (BP_*, BP_*BP) \to (E_*, E_*E)$$

that induces a functor

$$\Phi_*\colon BP_*BP\text{-comod} \to E_*E\text{-comod}.$$

defined by $\Phi_*M \cong E_* \otimes_{BP_*} M$. Landweber exactness implies that Φ_* is exact and

$$E_*X \cong \Phi_*(BP_*X)$$

for any spectrum X.

Since Φ_* commutes with all colimits, it ought to have a right adjoint, though this seems not to have been noticed before.

Lemma: The functor Φ_* has a right adjoint Φ^*.

Proof: One sees easily that

$$\Phi^*(E_*E \otimes_{E_*} M) \cong BP_*BP \otimes_{BP_*} M.$$

An arbitrary E_*E-comodule M fits into an exact sequence of comodules

$$0 \to M \xrightarrow{\psi} E_*E \otimes_{E_*} M \xrightarrow{\psi g} E_*E \otimes_{E_*} N$$

where $g\colon E_*E \otimes_{E_*} M \to N$ is the cokernel of ψ. It is not too difficult to define Φ^* on arbitrary maps of extended comodules, and then, since Φ^* will have to be left exact, we must define

$$\Phi^* M \cong \ker \Phi^*(\psi g).\Box$$

Now consider the counit of this adjunction

$$\epsilon\colon \Phi_*\Phi^*M \to M.$$

Plug in an extended comodule to find

$$\Phi_*\Phi^*(E_*E \otimes_{E_*} M) \cong \Phi_*(BP_*BP \otimes_{BP_*} M)$$
$$\cong E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} M \cong E_*E \otimes_{E_*} M.$$

Thus the counit ϵ is an isomorphism on extended E_*E-comodules. Since it is a natural transformation of left exact functors, and every comodule is a kernel of a map of extended comodules, we get the following theorem.

Theorem: The counit $\Phi_*\Phi^*M \to M$ is an isomorphism. In particular, Φ^* defines an equivalence of categories between E_*E-comodules and its image, a full subcategory of BP_*BP-comodules.

We now want to know: which subcategory of BP_*BP-comodules is it?

For this, we look at the other composite $\Phi^*\Phi_*$. One can easily see that this is a left exact idempotent functor, equipped with a natural transformation $M \to \Phi^*\Phi_* M$.

General nonsense tells us that $\Phi^*\Phi_*$ is a localization functor! We denote it by L_{E_*}; it is localization with respect to the class of maps $\mathcal{W} = \{f : \Phi^*\Phi_* f \text{ is an iso}\}$. This means that the map $M \to L_{E_*}M$ is in $\mathcal{W}$ and $L_{E_*}M$ is $\mathcal{W}$-local, which means that

$$BP_*BP\text{-comod}(f, L_{E_*}M)$$

is an isomorphism for all $f \in \mathcal{W}$.

Note that $\mathcal{W} = \{f : \Phi_* f \text{ is an iso}\}$ because Φ^* is an equivalence onto its image. But Φ_* is exact, so $\mathcal{W} = \{f : \ker f, \operatorname{coker} f \in \mathcal{A}_{E_*}\}$, where $\mathcal{A}_{E_*} = \ker \Phi_* = \{M : \Phi_* M = 0\}$.

We are now reduced to computing $\mathcal{A}_{E_*} = \ker \Phi_*$. Because Φ_* is left exact and commutes with colimits, $\mathcal{A}_{E_*}$ is a *hereditary torsion theory*. That is, it is closed under subobjects, quotient objects, extensions, and arbitrary direct sums (and suspensions, in the graded case).

There are some obvious hereditary torsion theories of BP_*BP-comodules. Let $\mathcal{T}_n$ be the full subcategory of v_n-torsion comodules, so that $\mathcal{T}_0$ is the p-torsion comodules and $\mathcal{T}_{-1}$ is all comodules. Take $\mathcal{T}_\infty = \{0\}$. One can easily check that the $\mathcal{T}_n$ are hereditary torsion theories, and by well-known results of Johnson-Yosimura,

$$\mathcal{T}_{-1} \supseteq \mathcal{T}_0 \supseteq \cdots \supseteq \mathcal{T}_n \supseteq \cdots .$$

Theorem: Suppose $\mathcal{T}$ is a hereditary torsion theory of BP_*BP-comodules, and suppose $\mathcal{T}$ contains a nonzero finitely presented comodule. Then $\mathcal{T} = \mathcal{T}_n$ for some $-1 \leq n < \infty$.

Let $I_n = (p, v_1, \ldots, v_{n-1})$ as usual, so $I_0 = 0$.

Theorem: Define $\operatorname{height} E$ to be the unique n with $E_*/I_n \neq 0$ and $E_*/I_{n+1} = 0$, or ∞ if $E_*/I_n \neq 0$ for all n. Then $\mathcal{A}_{E_*} = \mathcal{T}_{\operatorname{height} E}$.

Theorem: If E and E' are two Landweber exact commutative ring spectra of the same height, then the categories of E_*E-comodules and E'_*E'-comodules are equivalent.

We thus denote $L_{E(n)_*}$ by L_n, in analogy with the topological L_n. If E has height n, then the category of E_*E-comodules is equivalent to the category of L_n-local BP_*BP-comodules.

Corollary (Miller-Ravenel change of rings theorem): Suppose M and N are $v_n^{-1}BP_*(v_n^{-1}BP)$-comodules. Then

$$\text{Ext}^{**}_{v_n^{-1}BP_*(v_n^{-1}BP)}(M,N) \cong \text{Ext}^{**}_{E(n)_*E(n)}(E(n)_* \otimes_{BP_*} M, E(n)_* \otimes_{BP_*} N).$$

The Ext groups are isomorphic because the categories they are taken in are equivalent!

Can also recover change of rings theorems of Morava and Hovey-Sadofsky.

Corollary: Every nonzero E_*E-comodule has a nonzero primitive.

Because every BP_*BP-comodule does so, including the local ones.

Corollary: The primitives in E_*/I_k are $\mathbb{F}_p[v_k]$ if $k <$ height E and $\mathbb{F}_p[v_k, v_k^{-1}]$ if $k =$ height E.

This has to be interpreted correctly if $k = 0$.

Corollary: If I is an invariant radical ideal in E_*, then $I = I_n$ for some $n \leq$ height E. Converse is true for $E(n)_*$, E_{n*}, and $v_n^{-1}BP_*$.

Corollary: (Landweber filtration) Every E_*E-comodule M that is finitely presented over E_* admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t$$

by subcomodules so that each quotient M_i/M_{i-1} is isomorphic as a comodule to a suspension of E_*/I_k for some $k \leq$ height E, depending on i.

The functor L_n on BP_*BP-comodules is very interesting itself. Note that it is left exact, but has nontrivial right derived functors L_n^i.

Theorem: There is a strongly and conditionally convergent spectral sequence of comodules

$$E_2^{s,t} = (L_n^s BP_*X)_t \Rightarrow BP_{t-s}L_nX,$$

with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Every element in $E_2^{0,t}$ coming from BP_*X is a permanent cycle.

Theorem: (1) $L_n^s M = 0$ if $s > n$.

(2) $L_n BP_*/I_k = BP_*/I_k$ for $k < n$ but $L_n BP_*/I_n = v_n^{-1}BP_*/I_n$.

(3) $L_n^{n-k}(BP_*/I_k) = BP_*/(I_k, I_{n+1}^\infty)$, and all other derived functors are 0.

(4) $L_n^s M$ is the Cech cohomology, in the sense of Greenlees and May, of the BP_*-module M relative to I_{n+1}.