THE GHOST AND WEAK DIMENSIONS OF RINGS AND RING SPECTRA

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Abstract. The primary object of this paper is to prove the conjecture of [HL09a] explaining how to recover the weak dimension of a ring from its derived category. In the process, we develop a theory of weak dimension, which we call ghost dimension, for the generalized rings, known as ring spectra, that arise in algebraic topology.

Introduction

In a previous paper [HL09a], the authors considered the problem of recovering the weak dimension of a ring R from the derived category $\mathcal{D}(R)$, together with its distinguished object R. In that paper, the authors defined the **ghost dimension** of R, $\operatorname{gh.dim.} R$, and proved that $\operatorname{gh.dim.} R \geq \operatorname{w.dim.} R$, with equality holding when R is coherent or has weak dimension 1. In the present paper, we prove that $\operatorname{gh.dim.} R = \operatorname{w.dim.} R$ for all rings R.

The point of doing this, besides its intrinsic interest, is to allow consideration of weak dimension for more general kinds of rings. In algebraic topology, for example, there is a notion of a ring spectrum, or, more precisely, an S-algebra E [EKMM97]. Such an S-algebra has no elements in the usual sense. There is a category of (right) E-modules, but it is not abelian. However, there is a derived category $\mathcal{D}(E)$ of E, and it shares many of the formal properties of the derived category $\mathcal{D}(R)$ of an ordinary ring R; in particular, $\mathcal{D}(E)$ is a compactly generated triangulated category, and there are derived tensor products and derived Hom objects. In fact, every ring R has an associated Eilenberg-MacLane S-algebra HR, and $\mathcal{D}(HR)$ is equivalent to $\mathcal{D}(R)$. To define invariants of such S-algebras E, then, one way to proceed is to define usual ring invariants, such as the weak or global dimension, in terms of $\mathcal{D}(R)$, and then apply this definition to $\mathcal{D}(E)$ as well.

The second author did this for the (right) global dimension in his thesis, and we now summarize this. For further details, see [HL09a]. Define a map $f\colon X \to Y$ in $\mathcal{D}(E)$ to be a **ghost** if $\mathcal{D}(E)(E, f)_* = 0$. In the case that E is an ordinary ring R, a ghost is then just a map that induces the zero homomorphism on homology. If E is an S-algebra, a ghost is a map that induces the zero homomorphism on homotopy groups. The second author proved that the right global dimension of a ring R is the least n for which every composite of $n+1$ ghosts in $\mathcal{D}(R)$ is null, or ∞ if there are arbitrarily long nonzero composites of ghosts. We can then define the global dimension of an S-algebra E in an analogous fashion. The authors did this and investigated the S-algebras of global dimension 0 in [HL09b].

Weak dimension is more complicated, and there seem to be many possible definitions. A major goal of this paper is to elucidate the different possibilities and to

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find the correct one. To begin, we define a variant of global dimension where we restrict our attention to composites of ghosts emanating from a compact object. The **ghost dimension** of an S-algebra E or a ring R is the least n such that every composite

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+1}} X_{n+1}$$

of $n+1$ ghosts in $\mathcal{D}(E)$ (or $\mathcal{D}(R)$), where X_0 is a compact object, is null (or ∞ if there are arbitrarily long nonzero composites of ghosts out of compact objects). Recall that X is compact in a triangulated category $\mathcal{C}$ if the functor $\mathcal{C}(X, -)$ preserves coproducts. In particular, the compact objects of $\mathcal{D}(R)$ are the perfect complexes (complexes quasi-isomorphic to bounded complexes of finitely generated projectives), and the compact objects of $\mathcal{D}(E)$ are the retracts of finite cell E-modules. The ghost dimension of a ring was discussed in [HL09a], as mentioned above.

In addition, a version of weak dimension closely related to Rouquier's definition of the dimension of a triangulated category [Rou08] is defined in [HL09b]. Neither of these use the notion of a flat E-module. This obvious oversight was made because of a difficulty with flat modules that we now recall. If E is an S-algebra, and F_* is a flat left $E_* = \pi_* E = \mathcal{D}(E)(E, E)_*$-module, then we can form a homology theory (a coproduct-preserving exact functor to abelian groups) on $\mathcal{D}(E)$ that takes M to $M_* \otimes_{E_*} F_*$. One would like to say that Brown representability for homology theories then forces there to be a left E-module F with $\pi_* F \cong F_*$. Unfortunately, Brown representability does not hold for a general ring spectrum [Nee97, CKN01], so flat modules may not be realizable. This worrying phenomenon led us to doubt the utility of flat modules. However, we use them in this paper. One of the surprising things we discover is the following. Call $X \in \mathcal{D}(E)$ **flat** if X_* is a flat E_*-module. Then we prove that X is flat if and only if every ghost f whose domain is X is phantom, in the sense that $\mathcal{D}(E)(A, f) = 0$ for all compact $A \in \mathcal{D}(E)$. This gives us two different notions of flat dimension. The one most similar to the algebraic situation we call the **constructible flat dimension**, con. flat dim. X. It is a measure of how many steps one needs to construct X from flat objects of $\mathcal{D}(E)$. We reserve the term **flat dimension**, flat dim. X, for the smallest n such that every composite of $n+1$ ghosts with domain X is phantom. This seems algebraically strange, but has better properties. This gives several more notions of weak dimension: the maximal constructible flat (resp. flat) dimension of a compact E-module, and the maximal constructible flat (resp. flat) dimension of an arbitrary E-module. We show that all of these are equal to the ghost dimension, except possibly the maximal constructible flat dimension of an arbitrary E-module. The conjecture of [HL09a] that gh. dim. R = w. dim. R then follows.

In the end, we are left with three definitions of weak dimension for an S-algebra E. There is gh. dim. E, which coincides with the maximal flat dimension of any object. There is the maximal constructible flat dimension of any object, which agrees with gh. dim. E for $E = HR$, but possibly not in general. And there is the Rouquier dimension Rouq. dim. E, which agrees with gh. dim. E when E_* is coherent. We prove that the ghost dimension is right-left symmetric, which we have been unable to do with any of the other definitions. Hence we argue that the ghost dimension is the proper version of weak dimension for S-algebras E.

This subject sorely needs examples, in order to be sure that all these definitions are in fact distinct. It should be possible to find an ordinary ring R such that the Rouqier dimension of $\mathcal{D}(R)$ is distinct from the other dimensions. Such an example would involve serious analysis of the derived category of a non-coherent ring. To determine whether the constructible flat dimension is different from the flat dimension would seem to require a new idea.

Note that all modules we use in this paper are right modules unless explicitly stated otherwise. The reader who is interested only in ordinary rings can read R everywhere the symbol E or E_* appears, read "chain complex of R-modules" whenever the term "E-module" appears, and read H_*X everywhere X_* appears, for X an E-module.

1. Ghost dimension and Rouquier dimension

For an S-algebra E or a ring R, the authors have previously considered two different possible definitions related to weak dimension, which we now discuss. First of all, we can define the Rouquier dimension to be the maximum number of steps needed to build a compact object of $\mathcal{D}(E)$ from finitely many copies of E (along the lines of Rouquier [Rou08]). In more detail, given a class $\mathcal{A}$ of objects of $\mathcal{D}(E)$, define $\langle \mathcal{A} \rangle^n$ inductively as follows. Define $\langle \mathcal{A} \rangle^0$ to be the collection of all retracts of coproducts of suspensions of elements of $\mathcal{A}$, and define an object X to be in $\langle \mathcal{A} \rangle^n$ if and only if it is a retract of an object $\widetilde{X}$ for which there is an exact triangle

$$A \to Y \to \widetilde{X} \to \Sigma A$$

where $A \in \langle \mathcal{A} \rangle^0$, and $Y \in \langle \mathcal{A} \rangle^{n-1}$. If $\mathcal{A}$ is a class of compact objects, we define $\langle \mathcal{A} \rangle^n_f$ similarly, with $\langle \mathcal{A} \rangle^0_f$ being the collection of all retracts of *finite* coproducts of suspensions of elements of $\mathcal{A}$, and then using the same inductive procedure to define $\langle \mathcal{A} \rangle^n_f$. Then $\langle \mathcal{A} \rangle^n_f$ consists of compact objects in $\langle \mathcal{A} \rangle^n$, but there may be compact objects in $\langle \mathcal{A} \rangle^n$ that are nevertheless not in $\langle \mathcal{A} \rangle^n_f$.

We define the **Rouquier dimension** of E (or R), $\operatorname{Rouq.dim.} E$, to be the smallest n such that $\langle E \rangle^n_f$ is all of the compact objects, or ∞ if no such n exists. This was called the weak dimension in [HL09b], but that seems inappropriate, since we do not know that it agrees with the weak dimension when E is an ordinary ring R. We define the **ghost dimension** of E (or R), $\operatorname{gh.dim.} E$, to be the smallest n such that $\langle E \rangle^n$ contains all the compact objects. We also define the **projective dimension**, $\operatorname{proj.dim.} X$, of a given object X to be the smallest n such that $X \in \langle E \rangle^n$. This was called the **ghost length** in [HL09a]. Then $\operatorname{gh.dim.} E$ is the supremum of $\operatorname{proj.dim.} X$ for X compact.

Note that for any object X in $\mathcal{D}(E)$, X_* is a graded module over the graded ring E_*. We write $\operatorname{proj.dim.}_{E_*} X_*$ for its projective dimension. For example, when E is an ordinary ring R and X is an ordinary R-module M concentrated in one degree (or, equivalently, a projective resolution of M), then $\operatorname{proj.dim.}_{E_*} X_* = \operatorname{proj.dim.}_R M$ is the classical projective dimension of the R-module M. The following proposition explains the connection between the definition of projective dimension given in the previous paragraph and the definition given in the introduction.

Proposition 1.1. *Suppose E is an S-algebra or an ordinary ring, and $X \in \mathcal{D}(E)$. Then* $\operatorname{proj.dim.} X \le n$ *if and only if every composite of* $n+1$ *ghosts with domain X is the zero map. Furthermore,* $\operatorname{proj.dim.} X \le \operatorname{proj.dim.}_{E_*} X_*$*, with equality when* $\operatorname{proj.dim.} X = 0$ *and also when E is an ordinary ring and X is the projective resolution of a module M.*

This proposition is the content of Proposition 1.1, the proof of Proposition 1.3, and Lemma 1.4 of [HL09a], although Proposition 1.1 of [HL09a] is really due to Christensen [Chr98, Theorem 3.5].

We commonly call the objects P with $\operatorname{proj.dim.} P = 0$ **projective**, as this proposition implies P is projective if and only if P_* is a projective E_*-module. We note that the universal coefficient spectral sequence of [EKMM97, Theorem IV.4.1] implies that if P is projective then the natural map

$$\mathcal{D}(E)(P, X) \to \operatorname{Hom}_{E_*}(P_*, X_*)$$

is an isomorphism for all $X \in \mathcal{D}(E)$. The converse is also true, for if this natural map is an isomorphism, then there are no nonzero ghosts with domain P.

The following lemma gives the most obvious relationship between ghost dimension and Rouquier dimension.

Lemma 1.2. *Suppose E is an S-algebra or an ordinary ring. Then*

$$\operatorname{gh.dim.} E \leq \operatorname{Rouq.dim.} E,$$

with equality holding when $\operatorname{gh.dim.} E = 0$.

Proof. The inequality is clear. If $\operatorname{gh.dim.} E = 0$, then every compact object is a retract of a coproduct of suspensions of E, so is also a retract of a finite coproduct of suspensions of E. □

Note that S-algebras with $\operatorname{gh.dim.} E = 0$ are called **von Neumann regular**, because if R is an ordinary ring, $\operatorname{gh.dim.} R = 0$ if and only if R is von Neumann regular (see [HL09b]).

The ghost dimension and the Rouquier dimension agree when E_* is coherent, as we see in the following proposition.

Proposition 1.3. *Suppose E is an S-algebra or an ordinary ring for which E_* is coherent. Then $X \in \langle E \rangle_f^n$ if and only if $X \in \langle E \rangle^n$ and X is compact. Thus* $\operatorname{gh.dim.} E = \operatorname{Rouq.dim.} E$.

There is no reason to think that $\operatorname{gh.dim.} E = \operatorname{Rouq.dim.} E$ if E_* is not coherent, even if E is an ordinary ring R, but we do not know a counterexample.

Proof. We first prove the well-known fact that, since E_* is coherent, for every compact object X of $\mathcal{D}(E)$, X_* is a finitely presented E_*-module. Consider the class $\mathcal{C}$ of X for which X_* is a finitely presented E_*-module. We claim that $\mathcal{C}$ is a thick subcategory. Given this, since $\mathcal{C}$ contains E, it contains all the compact objects as required (this is well-known; see [HPS97, Theorem 2.1.3] for a general approach to this fact). To prove that $\mathcal{C}$ is thick, we must show that it is closed under suspensions and retracts, which is obvious in this case, and also if we have an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in which X and Y are in $\mathcal{C}$, then Z is in $\mathcal{C}$. Given such an exact triangle, Z_* is an extension of $\operatorname{coker} f_*$ by $\ker(\Sigma f_*)$. Finitely presented modules are always closed under cokernels of morphisms and extensions [Lam99, Lemma 4.54]. If E_* is coherent, finitely presented modules are closed under kernels of morphisms as well, and so Z_* is finitely presented. Indeed, if $g\colon M \to N$ is a morphism of finitely presented modules over a coherent ring, then $M/\ker f \cong \operatorname{im} f$ is a finitely generated submodule of the finitely presented module N, so is finitely presented. Hence $\ker f$

must be a finitely generated submodule of the finitely presented module M, so is finitely presented.

Now suppose $X \in \langle E\rangle^n$ and X is compact. By induction, because X_* is finitely presented, we can choose finite coproducts P_i of suspensions of E and exact triangles

$$X_{i+1} \to P_i \xrightarrow{f_i} X_i \xrightarrow{\delta_i} \Sigma X_{i+1}$$

where $X_0 = X$ and f_i is onto on homotopy, so δ_i is a ghost. Of course each X_i is compact. Then consider the exact triangle

$$\Sigma^i X_{i+1} \to Y_i \to X \to \Sigma^{i+1} X_{i+1},$$

where the last map is the composite of the maps $\Sigma^j X_j \to \Sigma^{j+1} X_{j+1}$. By applying the 3×3 lemma (well-known, but stated in [HPS97, Lemma A.1.2]) to the square

$$\begin{array}{ccc} X & \longrightarrow & \Sigma^i X_i \\ \| & & \downarrow \\ X & \longrightarrow & \Sigma^{i+1} X_{i+1} \end{array}$$

we see that there is an exact triangle

$$\Sigma^{i-1} P_i \to Y_{i-1} \to Y_i \to \Sigma^i P_i.$$

Hence Y_i is in $\langle E\rangle^i_f$. On the other hand, the map $X \to \Sigma^{n+1} X_{n+1}$ is the composite of $n+1$ ghosts, so it is null since $X \in \langle E\rangle^n$, using Proposition 1.1. Hence X is a retract of Y_n, so $X \in \langle E\rangle^n_f$. □

2. Flat dimension

We now offer a different approach to the weak dimension of an S-algebra or an ordinary ring using flat modules. As discussed in the introduction, we did not use these in [HL09a] because of the fundamental issue that homology functors are not always representable in $\mathcal{D}(E)$ for an S-algebra E, or even a ring R.

However, we can still define $\mathcal{F}$ to be the class of objects F in $\mathcal{D}(E)$ such that F_* is flat over E_*. In this case, we say that F is **flat** (as an object of $\mathcal{D}(E)$). We can then define an E-module $X \in \mathcal{D}(E)$ to have **constructible flat dimension** n, written $\text{con. flat dim.}\, X = n$, if $X \in \langle \mathcal{F}\rangle^n$. Note that $\text{con. flat dim.}\, X \leq \text{proj. dim.}\, X$, since every projective is flat. We can then consider the maximal constructible flat dimension of any object in $\mathcal{D}(R)$, or of just a compact object in $\mathcal{D}(R)$. Both of these are possible candidates for something like weak dimension. In principle, we could also consider a definition similar to the Rouquier dimension, using compact flat objects to resolve arbitrary compact objects, but we will see that a compact flat object is projective, so this would just recover Rouquier dimension.

Proposition 2.1. *We have* $\text{con. flat dim.}\, X \leq \text{flat dim.}\, X_*$ *for all* $X \in \mathcal{D}(E)$. *In particular, the maximal constructible flat dimension of an object in* $\mathcal{D}(E)$ *is bounded above by* $\text{w. dim.}\, E_*$.

Proof. There is nothing to prove if $\text{flat dim.}\, X_*$ is infinite, so suppose $\text{flat dim.}\, X_* = n$. Then by beginning a projective resolution of X_*, we get an exact sequence of E_*-modules

$$0 \to F \to P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} X_* \to 0$$

where each P_i is projective over E_* and F is flat over E_*. This gives us short exact sequences

$$0 \to K_{i+1} \to P_i \to K_i \to 0$$

for $i \leq n-1$, where $K_i = \ker d_{i-1}$, $K_0 = X_*$, and $K_n = F$. Because the P_i are projective, these short exact sequences are uniquely realizable by exact triangles in $\mathcal{D}(E)$

$$X_{i+1} \to Q_i \to X_i \to \Sigma X_{i+1}$$

where $X_0 = X$, $(X_i)_* = K_i$ and $(Q_i)_* = P_i$. In more detail, P_i is a retract of a direct sum of copies of E_*. Thus we can let Q_i be the corresponding retract of a coproduct of copies of E. Then one checks that a map out of Q_i to any object Y is equivalent to a map $P_i \to Y_*$. This gives us exact triangles of the form

$$\Sigma^i X_{i+1} \to Y_i \to X \to \Sigma^{i+1} X_{i+1}$$

for all i, where the last map is the composite of the maps $\Sigma^j X_j \to \Sigma^{j+1} X_{j+1}$. Applying the 3×3 lemma to the commutative square

$$\begin{array}{ccc} X & \longrightarrow & \Sigma^i X_i \\ \| & & \downarrow \\ X & \longrightarrow & \Sigma^{i+1} X_{i+1} \end{array}$$

gives us exact triangles

$$\Sigma^{i-1} Q_i \to Y_{i-1} \to Y_i \to \Sigma^i Q_i$$

for all i. In particular, con. flat dim. $Y_i \leq i$. Now the exact triangle

$$\Sigma^{n-1} X_n \to Y_{n-1} \to X \to \Sigma^n X_n$$

shows that con. flat dim. $X \leq n$, since $(X_n)_*$ is flat. □

We now give an alternative characterization of the flat objects in $\mathcal{D}(E)$. Recall that a **phantom** map in $\mathcal{D}(E)$ is a map $f\colon X \to Y$ such that $fg = 0$ for all $g\colon A \to X$ where A is compact. We need the following lemma.

Lemma 2.2. *Suppose E is an S-algebra. A map $f\colon X \to Y$ is phantom in $\mathcal{D}(E)$ if and only if $\pi_*(f \wedge_E Z) = 0$ for all left E-modules Z if and only if $\pi_*(f \wedge_E Z) = 0$ for all compact left E-modules Z.*

Proof. Spanier-Whitehead duality implies that $\pi_*(f \wedge_E Z) = 0$ for all compact left E-modules Z if and only if $\mathcal{D}(E)(W, f) = 0$ for all compact right E-modules W, which of course is the definition of f being phantom. It remains to show that, under this condition, $\pi_*(f \wedge_E Z) = 0$ for all left E-modules Z. But $\pi_*(- \wedge_E Z)$ is a homology theory on $\mathcal{D}(E)$, and phantom maps vanish on all homology theories [CS98, Proposition 1.1]. The reader should note that Christensen and Strickland are working in the ordinary category of spectra, where Spanier-Whitehead duality is internal, but the same proof will work for a general S-algebra E as long as we remember that Spanier-Whitehead duality shifts from left to right E-modules and vice versa. □

Proposition 2.3. *Suppose E is an S-algebra or a ring, and $X \in \mathcal{D}(E)$. Then the following are equivalent*:

(1) *X_* is a flat E_*-module.*
(2) *Every ghost with domain X is phantom.*

(3) *There is an exact triangle*

$$P \to X \xrightarrow{g} Y \to \Sigma P$$

where P is projective and g is phantom.

(4) *Every map $A \to X$, where A is compact, factors through a compact projective object.*

(5) *The natural map*

$$X_* \otimes_{E_*} Z_* \to \pi_*(X \wedge_E Z)$$

is an isomorphism for all left E-modules Z.

If E is an ordinary ring R, then $X \wedge_E Z$ would be the total left derived tensor product of the chain complex X of right R-modules and the chain complex Z of left R-modules.

Proof. For any X, there is an exact triangle

$$P \to X \xrightarrow{g} Y \to \Sigma P$$

in which P is projective and g is a ghost. Indeed, we simply take an epimorphism from a free E_*-module P_* to X_*. We then let P be the corresponding coproduct of suspensions of E, which is projective, and realize the map $P_* \to X_*$ as a map $P \to X$. The map g is then automatically a ghost, and every other ghost with domain X factors through g.

It follows from this that part (2) and (3) are equivalent. This also means that part (3) implies part (4), since part (3) means that a map $A \to X$, where A is compact, factors through a projective, and therefore a free E-module. Since A is compact, it must factor through a finite coproduct of suspensions of E.

We now show that part (4) implies part (1). Recall the filtered category $\Lambda(X)$ from [HPS97, Section 2.3] of maps from a compact object into X, and consider the full subcategory $\Lambda'(X)$ of maps from a compact projective into X. Given part (4), $\Lambda'(X)$ is cofinal in $\Lambda(X)$ and itself filtered. Thus, for any homology theory H, $H(X) = \operatorname{colim}_{\Lambda'(X)} H(P_\alpha)$ by [HPS97, Corollary 2.3.11]. In particular, X_* is a colimit of finitely generated projective modules, so is flat.

To see that part (1) implies part (5), use the universal coefficient spectral sequence

$$\operatorname{Tor}^{E_*}_{s,t}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z)$$

of [EKMM97, Theorem IV.4.1].

To see that part (5) implies part (2), suppose that g is a ghost with domain X. Part (5) then implies that $\pi_*(g \wedge_E Z) = 0$ for all left E-modules Z, which implies that g is phantom by Lemma 2.2. □

Recall that, for a general ring R, there are finitely generated flat modules which are not projective, though of course every finitely presented flat module is projective. The following corollary is an analog of this fact for S-algebras.

Corollary 2.4. *Suppose E is an S-algebra or an ordinary ring. If X is a compact flat object of $\mathcal{D}(E)$, then X is projective.*

Proof. The universal ghost out of X is phantom, and hence null. Thus X is projective, necessarily finitely generated since X is compact. □

We have been unable to fully generalize Proposition 2.3 to objects X with con. flat dim. $X = n$. We therefore make the following definition.

Definition 2.5. Suppose E is an S-algebra or an ordinary ring, and $X \in \mathcal{D}(E)$. We say that X has **flat dimension** at most n, written flat dim. $X \leq n$, if every composite of $n+1$ ghosts with domain X is phantom.

As we shall see, when E is an ordinary ring R, the maximal flat dimension of any object X is the same as the maximal flat dimension of any R-module (i.e., the classical weak dimension of R). In general, the maximal flat dimension of any object X is precisely the ghost dimension of E (Corollary 2.8). Since ghost dimension is left–right symmetric (Theorem 2.9) and agrees with the classical weak dimension when E is an ordinary ring, we argue that the ghost dimension is the proper version of weak dimension for S-algebras. We have the following theorem concerning flat dimension.

Theorem 2.6. *Suppose E is an S-algebra or an ordinary ring, and $X \in \mathcal{D}(E)$. Then* flat dim. $X \leq$ con. flat dim. X, *and the following are equivalent.*

(1) flat dim. $X \leq n$.
(2) *There is an exact triangle*
$$B \to X \xrightarrow{g} Y \to \Sigma B$$
where proj. dim. $B \leq n$ *and g is phantom.*
(3) *Every map $A \to X$, where A is compact, factors through a compact object B with* proj. dim. $B \leq n$.
(4) *For any left E-module Z, in the universal coefficient spectral sequence*
$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z),$$
we have $E^\infty_{s,*} = 0$ for all $s > n$.

It would be good to find an example where flat dim. $X <$ con. flat dim. X, or to prove they are always equal.

Proof. We show that con. flat dim. $X \leq n$ implies every composition of $n+1$ ghosts out of X is phantom by induction on n. The base case of $n = 0$ is Proposition 2.3. For the induction step, suppose con. flat dim. $X \leq n$,
$$X \xrightarrow{g_1} Z_1 \xrightarrow{g_2} \cdots \xrightarrow{g_{n+1}} Z_{n+1}$$
is the composition g of $n+1$ ghosts, and $f\colon A \to X$ is a map from a compact object. We must show $gf = 0$. Since con. flat dim. $X \leq n$, there is a cofiber sequence
$$F \to Y \xrightarrow{s} \widetilde{X} \xrightarrow{h} \Sigma F$$
where F is flat, con. flat dim. $Y \leq n-1$, and there are maps
$$X \xrightarrow{i} \widetilde{X} \xrightarrow{r} X$$
with $ri = 1_X$. Since $gf = (gr)(if)$, and gr is again a composition of $n+1$ ghosts, we can assume $X = \widetilde{X}$.

The composition hf factors through a finitely generated projective P, by Proposition 2.3. This gives us a commutative diagram

$$\begin{array}{ccccccc} \Sigma^{-1}P & \longrightarrow & B & \xrightarrow{t} & A & \xrightarrow{u} & P \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow \\ F & \longrightarrow & Y & \xrightarrow[s]{} & X & \xrightarrow[h]{} & \Sigma F \end{array}$$

whose rows are exact triangles. Note that B is necessarily a compact object, and so the composition $g_n \circ \cdots \circ g_1 s f'$ is null, since flat dim. $Y \leq n-1$. Hence we have

$$g_n \circ \cdots \circ g_1 f t = 0 \text{ so } g_n \circ \cdots \circ g_1 f = vu$$

for some map $v\colon P \to Z_n$. But then $g_{n+1}v = 0$, since P is projective, and so

$$g_{n+1} \circ \cdots \circ g_1 f = 0$$

as required. This completes the proof that flat dim. $X \leq$ con. flat dim. X.

The work of Christensen [Chr98, Theorem 3.5] implies that, for any X, there is an exact triangle

$$B \to X \xrightarrow{g} Y \to \Sigma B$$

with proj. dim. $B \leq n$ and g is a composite of $n+1$ ghosts. But then every composite of $n+1$ ghosts with domain X factors through g. Thus every composite of $n+1$ ghosts is phantom if and only if g is phantom, so part (1) and part (2) are equivalent.

Now, the universal coefficient spectral sequence for $\pi_*(X \wedge_E Z)$ is constructed as follows. Beginning with $X_0 = X$, we construct exact triangles

$$X_{i+1} \to Q_i \xrightarrow{h_i} X_i \xrightarrow{k_i} \Sigma X_{i+1}$$

as in the proof of Proposition 2.1, in which Q_i is projective, h_i is onto on homotopy, and k_i is a ghost. We then smash them with Z and take homotopy to get our spectral sequence of homological type. An element in $\pi_*(X \wedge_E Z)$ is detected in $E^\infty_{s,*}$ if and only if it is in the kernel of

$$\pi_*(X \wedge_E Z) \to \pi_*(\Sigma^j X_j \wedge_E Z)$$

for $j = s+1$ but not for $j = s$. Since the filtration is exhaustive, $E^\infty_{s,*} = 0$ for all $s > n$ if and only if the map

$$\pi_*(X \wedge_E Z) \to \pi_*(\Sigma^{n+1} X_{n+1} \wedge_E Z)$$

is zero for all Z. However, comparison of this construction of [EKMM97, Section IV.5] with [Chr98, Theorem 3.5] shows that in fact the map $X \to \Sigma^{n+1} X_{n+1}$ is the same as the universal composite of $n+1$ ghosts out of X, the map $g\colon X \to Y$ of the previous paragraph. Lemma 2.2 then shows part (2) and part (4) are equivalent.

It is clear that if every map from a compact to X factors through some object Y, compact or not, with proj. dim. $Y \leq n$, then every composite of $n+1$ ghosts out of X is phantom, so part (3) implies part (1). For the converse, in view of part (2), it suffices to prove that every map from a compact object to an object W with proj. dim. $W \leq n$ factors through a compact B with proj. dim. $B \leq n$. We proceed by induction on n. The base case $n = 0$ is implied by Proposition 2.3. For the induction step, suppose we have a map $f\colon A \to W$, where A is compact and $\operatorname{proj.dim.} W \leq n+1$. Then we have an exact triangle

$$P \xrightarrow{r} C \xrightarrow{s} \widetilde{W} \xrightarrow{t} \Sigma P$$

where P is projective, $\operatorname{proj.dim.} C \leq n$, and W is a retract of $\widetilde{W}$. It suffices to show that the composite

$$A \to W \to \widetilde{W}$$

factors through a compact B with $\operatorname{proj.dim.} B \leq n+1$. We can therefore assume $W = \widetilde{W}$. Since ΣP is projective and A is compact, we can write $tf = \phi h$ for some $h\colon A \to D$, where D is compact and projective. This gives us a map of exact triangles

$$\begin{array}{ccccccc}
\Sigma^{-1} D & \longrightarrow & Z & \xrightarrow{\kappa} & A & \xrightarrow{h} & D \\
{\scriptstyle \Sigma^{-1}\phi}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\
P & \underset{r}{\longrightarrow} & C & \underset{s}{\longrightarrow} & W & \underset{t}{\longrightarrow} & \Sigma P.
\end{array}$$

But then Z is necessarily compact, so by the induction hypothesis we can write $\psi = j\tau$, where the codomain V of τ is compact with $\operatorname{proj.dim.} V \leq n$. By taking the weak pushout (which amounts to applying the 3×3 lemma), we get the following diagram, whose rows are exact triangles:

$$\begin{array}{ccccccc}
\Sigma^{-1} D & \longrightarrow & Z & \xrightarrow{\kappa} & A & \xrightarrow{h} & D \\
\| & & {\scriptstyle \tau}\downarrow & & {\scriptstyle \sigma}\downarrow & & \| \\
\Sigma^{-1} D & \longrightarrow & V & \longrightarrow & V' & \longrightarrow & D \\
{\scriptstyle \Sigma^{-1}\phi}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \phi} \\
P & \underset{r}{\longrightarrow} & C & \underset{s}{\longrightarrow} & W & \underset{t}{\longrightarrow} & \Sigma P.
\end{array}$$

Now V' is a compact object with $\operatorname{proj.dim.} V' \leq n+1$, but unfortunately $\rho\sigma$ need not be equal to f. Nevertheless, we do have

$$\rho\sigma\kappa = s\psi = f\kappa$$

so $f - \rho\sigma = qh$ for some map $q\colon D \to W$. Altogether then, we have

$$f = \rho\sigma + qh.$$

This means that f factors through the compact object $V' \amalg D$. Indeed, f is the composite

$$A \xrightarrow{(\sigma,h)} V' \amalg D \xrightarrow{\rho+q} W.$$

Since $\operatorname{proj.dim.}(V' \amalg D) \leq n+1$, the proof is complete. □

Corollary 2.7. *Suppose E is an S-algebra or an ordinary ring. If X is a compact object of $\mathcal{D}(E)$, then*

$$\text{flat dim.}\, X = \text{con. flat dim.}\, X = \operatorname{proj.dim.} X.$$

In particular, $\operatorname{gh.dim.} E$ *is the maximal (constructible or not) flat dimension of a compact object of $\mathcal{D}(E)$, or ∞ if there is no such maximal dimension.*

Proof. We always have $\operatorname{flat\,dim.} X \leq \operatorname{con.\ flat\,dim.} X \leq \operatorname{proj.dim.} X$. Suppose $\operatorname{flat\,dim.} X = n$ and X is compact. Then every composition of $n+1$ ghosts out of X is phantom, hence null. Thus $\operatorname{proj.dim.} X \leq n$, so $\operatorname{proj.dim.} X = \operatorname{flat\,dim.} X$. □

Corollary 2.8. *Suppose E is an S-algebra or an ordinary ring. Then* $\operatorname{gh.dim.} E = \sup \operatorname{flat\,dim.} X$ *as X runs through arbitrary objects of $\mathcal{D}(E)$.*

We do not know whether this corollary remains true for the constructible flat dimension.

Proof. Corollary 2.7 implies that

$$\operatorname{gh.dim.} E = \sup_{X \text{ compact}} \operatorname{flat\,dim.} X \leq \sup_{X \text{ arbitrary}} \operatorname{flat\,dim.} X.$$

Now suppose $\operatorname{gh.dim.} E = n$, so that $\operatorname{flat\,dim.} F \leq n$ for all compact objects F of $\mathcal{D}(E)$. Choose an arbitrary $X \in \mathcal{D}(E)$, and consider the universal coefficient spectral sequence

$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(X_*, Z_*) \Rightarrow \pi_{t-s}(X \wedge_E Z)$$

for an arbitrary left E-module Z. We must show that $E^\infty_{s,*} = 0$ for $s > n$. Since this spectral sequence is of homological type, this means we must show that every element in $\pi_*(X \wedge_E Z)$ is in filtration n (and possibly lower filtration as well). But the functor $\pi_*(- \wedge_E Z)$ is a homology theory on right E-modules, so

$$\pi_*(X \wedge_E Z) = \operatorname{colim} \pi_*(F \wedge_E Z),$$

where the colimit is taken over all maps $F \to X$ from a compact object to X. Then any element of $\pi_*(X \wedge_E Z)$ comes from some $\pi_*(F \wedge_E Z)$, where it lies in filtration n. Naturality of the spectral sequence implies that it also lies in filtration n in $\pi_*(X \wedge_E Z)$. □

We now point out another advantage of the ghost dimension; it is left–right symmetric, like the usual weak dimension of rings.

Theorem 2.9. *Suppose E is an S-algebra or an ordinary ring. Then*

$$\operatorname{gh.dim.} E = \operatorname{gh.dim.} E^{\mathrm{op}}.$$

Proof. The ghost dimension of E is the largest n such that there exists a right E-module X and a left E-module Y for which $E^\infty_{n,*}$ is nonzero in the universal coefficient spectral sequence for $\pi_*(X \wedge_E Y)$. This is obviously left-right symmetric. □

Summing up, then, we are left with three possible definitions for the weak dimension of an S-algebra E. We list the basic inequalities between these definitions in the following theorem, which also proves the main conjecture of [HL09a] that $\operatorname{gh.dim.} R = \operatorname{w.dim.} R$ for ordinary rings R.

Theorem 2.10. *Suppose E is an S-algebra or an ordinary ring. Then we have*

$$\begin{aligned}\operatorname{gh.dim.} E &= \sup_{X \text{ compact}} \operatorname{con.\ flat\,dim.} X = \sup_{X \text{ compact}} \operatorname{proj.dim.} X \\ &= \sup_{X \text{ compact}} \operatorname{flat\,dim.} X = \sup_{X \text{ arbitrary}} \operatorname{flat\,dim.} X.\end{aligned}$$

Furthermore,

$$\operatorname{gh.dim.} E \leq \sup_{X \text{ arbitrary}} \operatorname{con.\ flat\,dim.} X \leq \operatorname{w.dim.} E_*$$

with equality if E is an ordinary ring. Finally,

$$\mathrm{gh.\,dim.}\, E \leq \mathrm{Rouq.\,dim.}\, E$$

with equality if E_* is coherent.

Proof. The only thing we have not already proved is that equality holds in the first chain of inequalities when R is an ordinary ring. But the main result of [HL09a] is that $\mathrm{w.\,dim.}\, R \leq \mathrm{gh.\,dim.}\, R$, giving us the desired equalities. □

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