Additive closed symmetric monoidal structures on R-modules

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Abstract

In this paper, we classify additive closed symmetric monoidal structures on the category of left R-modules by using Watts' theorem. An additive closed symmetric monoidal structure is equivalent to an R-module $\Lambda_{A,B}$ equipped with two commuting right R-module structures represented by the symbols A and B, an R-module K to serve as the unit, and certain isomorphisms. We use this result to look at simple cases. We find rings R for which there are no additive closed symmetric monoidal structures on R-modules, for which there is exactly one (up to isomorphism), for which there are exactly seven, and for which there are a proper class of isomorphism classes of such structures. We also prove some general structural results; for example, we prove that the unit K must always be a finitely generated R-module.

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Introduction

It is well known that the category of left R-modules becomes closed symmetric monoidal under the tensor product $A \otimes_R B$ if and only if R is commutative. However, there are many other cases when the category of R-modules is closed symmetric monoidal. For example, if k is a field and G is a group, the category of $k[G]$-modules (that is, representations of G on k-vector spaces) is closed symmetric monoidal under $A \otimes_k B$, even though $k[G]$ is not commutative in general. This is explained by the fact that $k[G]$ is a Hopf algebra. But there are other examples where R is not a Hopf algebra, such as the category of perverse R-modules considered in [Hov09].

So a natural question to ask is just what one needs to know about R in order to produce a closed symmetric monoidal structure on the category of R-modules. Of course, we do not really want an arbitrary closed symmetric

monoidal structure; we require that the monoidal product be an additive functor in both variables. We would like to be able to answer basic questions such as the following. Are there rings R where the category of R-modules cannot be given an additive closed symmetric monoidal structure? Are there rings R where the category of R-modules possesses a unique additive closed symmetric monoidal structure?

At first glance, such problems seem completely intractable because closed symmetric monoidal structures are so complicated, involving the entire category of R-modules. The key ingredient, though, is Watts' theorem [Wat60]. This theorem says that any additive functor F from R-modules to R-modules that is right exact and commutes with direct sums is naturally isomorphic to $\Lambda \otimes_R (-)$ for some R-bimodule Λ. After some work, we then see in Theorem 2.1 that a closed symmetric monoidal structure $A \wedge B$ on left R-modules must be given by

$$A \wedge B \cong ((R \wedge R) \otimes_R A) \otimes_R B,$$

so that the functor $- \wedge -$ is determined by $R \wedge R$, as a 2-fold bimodule (one left module structure, and two right module structures).

Then the natural thing to do is try to determine which 2-fold bimodules $\Lambda_{A,B}$ actually arise as $R \wedge R$ for some closed symmetric monoidal structure $- \wedge -$. This is more complicated than it seems because one must deal with the coherence isomorphisms of a closed symmetric monoidal structure, but of course it can be done; see Theorem 2.3. We also determine in Theorem 2.4 (and the discussion following it) when two symmetric monoidal structures determined by Λ and Λ' are equivalent. This involves an isomorphism

$$\Lambda' \otimes X \otimes X \to X \otimes \Lambda$$

of 2-fold bimodules, where X is an element of the bimodule Picard group of R.

We establish some basic structural results, though we think there is much more to say. For example, we show in Theorem 4.4 and Theorem 4.5 that the unit K of an additive closed symmetric monoidal structure on left R-modules must be a finitely generated R-module with a commutative endomorphism ring. To proceed further along these lines, it might be worthwhile to develop a theory of flatness for an additive symmetric monoidal structure $\wedge$ on R-modules, and concentrate on those additive symmetric monoidal structures for which projectives are flat.

We also consider examples. For example, if R is a field (Theorem 3.3) or a principal ideal domain that does not contain a field (Theorem 4.9), then there is exactly one additive closed symmetric monoidal structure on R-modules (up to symmetric monoidal equivalence). We also show in Theorem 3.3 that if R is a division ring that is not a field, there are no additive closed symmetric monoidal structures on R-modules. If R is the group ring $\mathbb{F}_2[\mathbb{Z}/2]$, there are precisely 7 different closed symmetric monoidal structures on R-modules, though only three different underlying functors (see the comments after Theorem 3.6). If R is the group ring $k[\mathbb{Z}/2]$ where the characteristic is not 2, however, there are a proper class of inequivalent additive closed symmetric monoidal structures on

the category of R-modules (Theorem 3.15). Most of these cannot come from Hopf algebra structures on R.

Throughout this paper, the symbol $\otimes$ will denote the tensor product over the ring R unless otherwise stated. Furthermore, all functors will be assumed to be additive, even if not explicitly stated to be so.

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1. n-fold bimodules

Throughout this paper, we will be working with the category of left R-modules, but we will frequently have to work with left R-modules that have multiple different commuting right R-module structures. This necessitates some complicated notation. We will denote an $R \otimes_{\mathbb{Z}} (R^{\mathrm{op}})^{\otimes_{\mathbb{Z}} n}$-module by $\Lambda_{1,2,\ldots,n}$, where the subscripts denote the commuting right R-actions. If we need elements, we will write $x \odot_i r$ for the ith multiplication. We will call such an object an n-fold bimodule, and denote the category of such things as $\mathrm{Bimod}_n(R)$. Now the symmetric group Σ_n acts on $\mathrm{Bimod}_n(R)$ by permuting the right module structures. Indeed, if $\sigma \in \Sigma_n$ is a permutation, then

$$(\sigma\Lambda)_{1,2,\ldots,n} = \Lambda_{\sigma(1),\ldots,\sigma(n)}.$$

We note that in practice, as we will see below, it is usually easier to denote the different right module structures by letters, such as $\Lambda_{A,B,C}$. If σ denotes the permutation (132), for example, then we would write $\sigma\Lambda$ as $\Lambda_{C,A,B}$. A map

$$f\colon \Lambda_{A,B,C} \to \Lambda_{C,A,B}$$

would then be a map of left R-modules such that $f(x \odot_1 r) = f(x) \odot_2 r$ (matching up the position of the A's), $f(x \odot_2 r) = f(x) \odot_3 r$, and $f(x \odot_3 r) = f(x) \odot_1 r$.

Now, note that the tensor product over R defines a bifunctor

$$\mathrm{Bimod}_n(R) \times \mathrm{Bimod}_m(R) \to \mathrm{Bimod}_{n+m-1}(R),$$

obtained by tensoring the nth right module structure on the first factor with the left module structure on the second factor. For example, we will have expressions like

$$f\colon \Lambda_{C,\Gamma} \otimes \Gamma_{B,A} \to \Lambda_{\Gamma,A} \otimes \Gamma_{C,B}.$$

This denotes a map of 3-fold bimodules. In the domain, we tensor the second right R-module structure on Λ with the left module structure on Γ, whereas in the range we tensor the first bimodule structure on Λ with the left module structure on Γ. Furthermore, we have $f((x \odot_1 r) \otimes y) = f(x \otimes y) \odot_2 r$, where the second right module structure on the range comes from the first right module structure on Γ. Similarly, we have $f(x \otimes (y \odot_1 r)) = f(x \otimes y) \odot_3 r$. We also have $f(x \otimes (y \odot_2 r)) = f(x \otimes y) \odot_1 r$, where the first right module structure on the range comes from the **second** right module structure on Λ, because we have used the first one to form the tensor product in the range.

3

2. Closed symmetric monoidal structures

In this section, we prove our main classification result. We remind the reader that all tensor products are over R unless otherwise stated.

Theorem 2.1. *Suppose the category of R-modules admits an additive closed symmetric monoidal structure $- \wedge -$. Then $R \wedge R$ is a 2-fold bimodule, and there is a natural isomorphism of bifunctors*

$$(R \wedge R)_{B,A} \otimes A \otimes B \cong A \wedge B.$$

This theorem is written using the notation of the previous section, so that, in the domain of this isomorphism

$$(x \odot_1 r) \otimes y \otimes z = x \otimes y \otimes (rz) \quad \text{and} \quad (x \odot_2 r) \otimes y \otimes z = x \otimes (ry) \otimes z.$$

Also note that if R is a k-algebra, for a commutative ring k, we can look at k-linear closed symmetric monoidal structures. This would mean that multiplication by $x \in k$ on A would induce multiplication by x on $A \wedge B$ and on $B \wedge A$ for all B. In this case, the three potentially different actions of k on $R \wedge R$ would in fact all be the same.

Proof. Fix an R-module A, and consider the functor $B \mapsto A \wedge B$ from left R-modules to left R-modules. This functor preserves direct sums and is right exact (it is a left adjoint, because of the closed structure). Watt's theorem [Wat60] then implies that $A \wedge R$ is an R-bimodule, and

$$(A \wedge R) \otimes B \cong A \wedge B$$

naturally in B. To see that this is also natural in A, recall that the map

$$\alpha_{A,B} \colon (A \wedge R) \otimes B \to A \wedge B$$

is defined as follows, in the proof of Watt's theorem. Given $b \in B$, let $\phi_b \colon R \to B$ denote the map of R-modules that takes 1 to b. There is then an induced map $A \wedge \phi_b$, and $\alpha_{A,B}(x \otimes b)$ is defined to be $(A \wedge \phi_b)(x)$. From this, it is easy to check that $\alpha_{A,B}$ is natural in A as well.

Now let $G(A) = A \wedge R$. Then $A \mapsto G(A)$ is a right exact functor from R-modules to R-bimodules (or $R \otimes_{\mathbb{Z}} R^{\mathrm{op}}$-modules) that preserves direct sums. We can therefore apply Watt's theorem [Wat60] again to give us the desired result. $\square$

The natural question then arises as to which 2-fold bimodules $\Lambda_{B,A}$ define a closed symmetric monoidal structure on the category of R-modules.

We first point out that the closed structure always exists.

Lemma 2.2. *Suppose $\Lambda_{B,A}$ is a 2-fold R-bimodule. Then we have a natural isomorphism*

$$R\text{-mod}(\Lambda_{B,A} \otimes A \otimes B, P) \cong R\text{-mod}(A, \mathrm{Bimod}(\Lambda_{B,A}, \mathbb{Z}\text{-mod}(B, P))),$$

4

where we use the right R-module structure on Λ denoted by the subscript B to form $\mathrm{Bimod}(\Lambda_{B,A}, \mathbb{Z}\text{-mod}(B,P))$, *and we use the one denoted by the subscript A to make this abelian group into a left R-module.*

Here, as usual, $\mathbb{Z}\text{-mod}(B,P)$ is an R-module via the R-action on P, and a right R-module via the R-action on B.

Proof. This is really just an exercise in adjointness of tensor and Hom, though one has to be careful to keep track of all the actions. It is easiest to work more generally. Suppose M is a bimodule. Then we have a natural isomorphism

$$\phi\colon R\text{-mod}(M \otimes B, P) \cong \mathrm{Bimod}(M, \mathbb{Z}\text{-mod}(B,P)).$$

This isomorphism is defined as usual by $\phi(f)(m)(b) = f(m \otimes b)$. The reader must check that $\phi(f)$ is a map of bimodules. To see that ϕ is an isomorphism, one constructs its inverse ψ, where $\psi(g)(m \otimes b) = g(m)(b)$. Again, there are many details to check, which we leave to the reader. Applying this isomorphism to $M = \Lambda_{B,A} \otimes A$, we get

$$R\text{-mod}(\Lambda_{B,A} \otimes A \otimes B, P) \cong \mathrm{Bimod}(\Lambda_{B,A} \otimes A, \mathbb{Z}\text{-mod}(B,P)).$$

Now suppose N is a general bimodule. Then there is a natural isomorphism

$$\sigma\colon \mathrm{Bimod}(\Lambda_{B,A} \otimes A, N) \to R\text{-mod}(A, \mathrm{Bimod}(\Lambda_{B,A}, N)).$$

Once again, we have $\sigma(f)(a)(\lambda) = f(\lambda \otimes a)$, and, for the inverse τ of σ, we have $\tau(g)(\lambda \otimes a) = g(a)(\lambda)$. We leave it to the reader to check the details. Taking $N = \mathbb{Z}\text{-mod}(B,P)$ completes the proof. $\qquad\square$

Naturally, the other conditions necessary for a symmetric monoidal structure are considerably more complicated. The basic idea, however, is simple. In order to get symmetry of the product $A \wedge B$, we need the two right module structures on Λ to be isomorphic. In order to get associativity of $A \wedge B$, we need the three different right module structures on $\Lambda \otimes \Lambda$ to be isomorphic. These two together, of course, will imply that all of the different permutations of the $n+1$ different right module structures on $\Lambda^{\otimes n}$ will be isomorphic. Then we also need a unit.

Theorem 2.3. *Let $\Lambda_{B,A}$ be a 2-fold bimodule used to define $- \wedge -$ on R-modules. There is a one-to-one correspondence between additive closed symmetric monoidal structures on R-modules with $- \wedge -$ as the monoidal product and the following data*:

(a) *An associativity isomorphism of 3-fold bimodules*

$$a\colon \Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \to \Lambda_{\Lambda,A} \otimes \Lambda_{C,B}.$$

This can be remembered by noting that the subscripts on the first Λ in the target are the second subscripts on the two Λ's in the domain, and the subscripts on the second Λ in the target are the first subscripts on the two Λ's in the domain.

(b) A left R-module K and a unit isomorphism $\ell\colon \Lambda_{B,K} \otimes K \cong R_B$ of bimodules.

(c) A commutativity isomorphism $c\colon \Lambda_{B,A} \to \Lambda_{A,B}$.

This data must satisfy the following coherence conditions.

1. *(Associativity pentagon) Let $\Gamma = \Delta = \Lambda$ for notational clarity. Then the following diagram commutes.*

$$\Lambda_{D,\Gamma} \otimes \Gamma_{C,\Delta} \otimes \Delta_{B,A} \xrightarrow{a\otimes 1} \Lambda_{\Gamma,\Delta} \otimes \Gamma_{D,C} \otimes \Delta_{B,A} \xrightarrow{1\otimes T} \Lambda_{\Delta,\Gamma} \otimes \Gamma_{B,A} \otimes \Delta_{D,C}$$

$$\downarrow{\scriptstyle 1\otimes a} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle a\otimes 1}$$

$$\Lambda_{D,\Gamma} \otimes \Gamma_{\Delta,A} \otimes \Delta_{C,B} \xrightarrow[a\otimes 1]{} \Lambda_{\Gamma,A} \otimes \Gamma_{D,\Delta} \otimes \Delta_{C,B} \xrightarrow[1\otimes a]{} \Lambda_{\Gamma,A} \otimes \Gamma_{\Delta,B} \otimes \Delta_{D,C}$$

Here $1\otimes T$ switches the last two factors using the commutativity isomorphism of $\otimes_R$, but also reverses the symbols Γ and Δ, which after all both mean Λ. This necessitates changing the subscripts on Λ as well.

2. *(Compatibility of left and right unit) The following diagram commutes:*

$$\Lambda_{B,\Lambda} \otimes \Lambda_{K,A} \otimes K \xrightarrow{a\otimes 1} \Lambda_{\Lambda,A} \otimes \Lambda_{B,K} \otimes K \xrightarrow{1\otimes\ell} \Lambda_{B,A}$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\Lambda_{B,\Lambda} \otimes \Lambda_{K,A} \otimes K \xrightarrow[1\otimes c\otimes 1]{} \Lambda_{B,\Lambda} \otimes \Lambda_{A,K} \otimes K \xrightarrow[1\otimes\ell]{} \Lambda_{B,A}.$$

3. *(Commutativity-associativity hexagon) The following diagram commutes.*

$$\Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \xrightarrow{1\otimes c} \Lambda_{C,\Lambda} \otimes \Lambda_{A,B} \xrightarrow{a} \Lambda_{\Lambda,B} \otimes \Lambda_{C,A} \xrightarrow{1\otimes c} \Lambda_{\Lambda,B} \otimes \Lambda_{A,C}$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \xrightarrow[a]{} \Lambda_{\Lambda,A} \otimes \Lambda_{C,B} \xrightarrow[c\otimes 1]{} \Lambda_{A,\Lambda} \otimes \Lambda_{C,B} \xrightarrow[a]{} \Lambda_{\Lambda,B} \otimes \Lambda_{A,C}.$$

4. *The composite*

$$\Lambda_{B,A} \xrightarrow{c} \Lambda_{A,B} \xrightarrow{c} \Lambda_{B,A}$$

   is the identity.

If R is a k-algebra for a commutative ring k, and we are looking at k-linear closed symmetric monoidal structures, then this theorem remains true as long as the three different k-module structures on Λ are the same.

Proof. Just using the usual associativity and commutativity isomorphisms for $\otimes_R$, we find natural isomorphisms

$$(A \wedge B) \wedge C \cong \Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \otimes A \otimes B \otimes C$$

and

$$A \wedge (B \wedge C) \cong \Lambda_{\Lambda,A} \otimes \Lambda_{C,B} \otimes A \otimes B \otimes C.$$

Given a, it is now clear how to define a natural associativity isomorphism $a_{A,B,C}$ for $-\wedge-$, simply as $a \otimes 1 \otimes 1 \otimes 1$. On the other hand, given the natural

associativity isomorphism $a_{A,B,C}$, we let $A = B = C = R$ to get a. One can see that a then respects the given right module structures by using naturality with respect to the right multiplication by x maps $r_x \colon R \to R$, in the A, B, and C slots.

There is a similar equivalence between the isomorphism $\ell \colon \Lambda_{B,K} \otimes K \to R_B$ and a natural left unit isomorphism $\ell_B \colon K \wedge B \to B$. There is also a similar equivalence between c and a natural commutativity isomorphism $c_{A,B}$.

An excellent reference for the coherence diagrams needed to make $a_{A,B,C}$, ℓ_B, and $c_{A,B}$ part of a symmetric monoidal structure is [JS93], particularly Propositions 1.1 and 2.1. They show that the only coherence diagrams needed are the associativity pentagon, the compatibility between the right and left unit, the commutativity-associativity hexagon (diagram B1 of [JS93, p. 33]), the fact that the right unit is $r_B = \ell_B c_{B,K}$, and the fact that c^2 is the identity. Given $c_{A,B}$, this means we do not need r_B, so we have omitted it. One must now merely translate these coherence diagrams into analogous facts about a, ℓ, and c to complete the proof.

The associativity pentagon is perhaps the most confusing, so we will discuss that one in some detail, and leave the others to the reader. Here is the standard associativity pentagon.

$$((A \wedge B) \wedge C) \wedge D \xrightarrow{a_{A \wedge B, C, D}} (A \wedge B) \wedge (C \wedge D) \xrightarrow{a_{A,B,C \wedge D}} A \wedge (B \wedge (C \wedge D))$$

$$a_{A,B,C} \wedge 1 \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$(A \wedge (B \wedge C)) \wedge D \xrightarrow[a_{A, B \wedge C, D}]{} A \wedge ((B \wedge C) \wedge D) \xrightarrow[1 \wedge a_{B,C,D}]{} A \wedge (B \wedge (C \wedge D))$$

Using the standard commutativity and associativity isomorphisms of $\otimes$, the first term

$$((A \wedge B) \wedge C) \wedge D$$

is represented by $\Lambda_{D,\Gamma} \otimes \Gamma_{C,\Delta} \otimes \Delta_{B,A}$ (tensored with $A \otimes B \otimes C \otimes D$). Here $\Delta_{B,A}$ tensors A and B, $\Gamma_{C,\Delta}$ tensors $(A \wedge B)$ and C, and $\Lambda_{D,\Gamma}$ tensors $((A \wedge B) \wedge C)$ and D. The map $a_{A \wedge B, C, D}$ treats $A \wedge B$ as a single object, and will therefore leave the factor $\Delta_{B,A}$ unchanged, so is represented by

$$a \otimes 1 \colon \Lambda_{D,\Gamma} \otimes \Gamma_{C,\Delta} \otimes \Delta_{B,A} \to \Lambda_{\Gamma,\Delta} \otimes \Gamma_{D,C} \otimes \Delta_{B,A}.$$

The map $a_{A,B,C \wedge D}$ treats $C \wedge D$ as a single object, and will therefore leave $\Gamma_{D,C}$ unchanged and apply a to the other two. In order to do this, we would like to switch the order of $\Gamma_{D,C}$ and $\Delta_{B,A}$, and then apply $a \otimes 1$. It turns out to be notationally much easier later if we also reverse the names of Γ and Δ, so that the map $a_{A,B,C \wedge D}$ is represented by the composite

$$\Lambda_{\Gamma,\Delta} \otimes \Gamma_{D,C} \otimes \Delta_{B,A} \xrightarrow{1 \otimes T} \Lambda_{\Delta,\Gamma} \otimes \Gamma_{B,A} \otimes \Delta_{D,C} \xrightarrow{a \otimes 1} \Lambda_{\Gamma,A} \otimes \Gamma_{\Delta,B} \otimes \Delta_{D,C}.$$

This completes the clockwise half of the associativity pentagon. The counterclockwise part is simpler. The first map $a_{A,B,C} \wedge 1$ leaves D alone, so will leave $\Lambda_{D,\Gamma}$ alone. It is therefore represented by

$$1 \otimes a \colon \Lambda_{D,\Gamma} \otimes \Gamma_{C,\Delta} \otimes \Delta_{B,A} \to \Lambda_{D,\Gamma} \otimes \Gamma_{\Delta,A} \otimes \Delta_{C,B}.$$

The next map $a_{A,B\wedge C,D}$ treats $B\wedge C$ as a single entity, so will leave $\Delta_{C,B}$ alone. It is then represented by

$$a \otimes 1\colon \Lambda_{D,\Gamma} \otimes \Gamma_{\Delta,A} \otimes \Delta_{C,B} \to \Lambda_{\Gamma,A} \otimes \Gamma_{D,\Delta} \otimes \Delta_{C,B}.$$

Finally, the last map $1 \otimes a_{B,C,D}$ leaves A alone, so will leave $\Lambda_{\Gamma,A}$ alone. It is reprensented by

$$a \otimes 1\colon \Lambda_{\Gamma,A} \otimes \Gamma_{D,\Delta} \otimes \Delta_{C,B} \to \Lambda_{\Gamma,A} \otimes \Gamma_{\Delta,B} \otimes \Delta_{D,C}.$$

This completes the construction of the associativity pentagon. $\qquad\square$

We now point out that Watt's theorem can also be used to classify additive symmetric monoidal equivalences between additive symmetric monoidal structures on R-modules. In an attempt to make the various bimodule structures clear, we have used Y, Z, and W as alternative names for X in the theorem below. We have also used T for the usual commutativity isomorphism of the tensor product and for a general permutation of tensor factors.

Theorem 2.4. *Suppose $\wedge$ and $\square$ are additive symmetric monoidal structures on the category of R-modules with units K and K', respectively, and represented by the 2-fold bimodules Λ and Λ', respectively. Then an additive symmetric monoidal functor from $\wedge$ to $\square$ that has a right adjoint is equivalent to a bimodule X, an isomorphism $\eta\colon K' \to X \otimes K$, and an isomorphism*

$$m\colon \Lambda'_{Y,X} \otimes X_A \otimes Y_B \to X \otimes \Lambda_{B,A}$$

of 2-fold bimodules, such that the following diagrams commute:

1. *(Unit)*

$$
\begin{array}{ccc}
\Lambda'_{Y,K'} \otimes K' \otimes Y_B & \xrightarrow{\ 1\otimes\eta\otimes1\ } & \Lambda'_{Y,X} \otimes X_K \otimes K \otimes Y_B \\
{\scriptstyle \ell\otimes1}\downarrow & & \downarrow{\scriptstyle 1\otimes1\otimes T} \\
Y_B & & \Lambda'_{Y,X} \otimes X_K \otimes Y_B \otimes K \\
\| & & \downarrow{\scriptstyle m\otimes1} \\
X_B & \xleftarrow[\ 1\otimes\ell\ ]{} & X \otimes \Lambda_{B,K} \otimes K
\end{array}
$$

2. *(Commutativity)*

$$
\begin{array}{ccc}
\Lambda'_{Y,X} \otimes X_A \otimes Y_B & \xrightarrow{\ m\ } & X \otimes \Lambda_{B,A} \\
{\scriptstyle c\otimes T}\downarrow & & \downarrow{\scriptstyle 1\otimes c} \\
\Lambda'_{X,Y} \otimes Y_B \otimes X_A & \xrightarrow[\ m\ ]{} & X \otimes \Lambda_{A,B}
\end{array}
$$

8

3. *(Associativity)*

$$\Lambda'_{Z,\Lambda'} \otimes \Lambda'_{Y,X} \otimes X_A \otimes Y_B \otimes Z_C \xrightarrow{\;1\otimes m\otimes 1\;} \Lambda'_{Z,X} \otimes (X_\Lambda \otimes \Lambda_{B,A}) \otimes Z_C$$

$$a\otimes T \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow T$$

$$\Lambda'_{\Lambda',X} \otimes \Lambda'_{Z,Y} \otimes Y_B \otimes Z_C \otimes X_A \qquad\qquad \Lambda'_{Z,X} \otimes X_\Lambda \otimes Z_C \otimes \Lambda_{B,A}$$

$$1\otimes m\otimes 1 \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow m\otimes 1$$

$$\Lambda'_{W,X} \otimes (W \otimes \Lambda_{C,B}) \otimes X_A \qquad\qquad X \otimes \Lambda_{C,\Lambda} \otimes \Lambda_{B,A}$$

$$T \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 1\otimes a$$

$$\Lambda'_{W,X} \otimes X_A \otimes W_\Lambda \otimes \Lambda_{C,B} \xrightarrow[\;m\otimes 1\;]{} X \otimes \Lambda_{\Lambda,A} \otimes \Lambda_{C,B}$$

Composition of additive symmetric monoidal functors corresponds to the tensor product of bimodules, and the identity functor corresponds to the bimodule R_R. Thus, additive symmetric monoidal equivalences of additive symmetric monoidal structures are given by tensoring with a bimodule that lies in the bimodule Picard group (see [Yek99]). In fact, if X lies in the bimodule Picard group, then tensoring with X loses no information. In this case, then, the compatibility diagrams above show that the isomorphisms ℓ, c, and a for $\square$ are determined by the corresponding isomorphisms for $\wedge$, together with m and η. Thus we can think of the bimodule Picard group as acting on symmetric monoidal structures with fixed unit K, though there is also an action by the automorphisms of K, and, if Λ is fixed, the 2-fold bimodule automorphisms of Λ.

One could similarly prove that natural transformations between additive symmetric monoidal functors represented by X_1 and X_2 are induced by maps of bimodules $X_1 \to X_2$.

Proof. Suppose F is a symmetric monoidal functor from $\wedge$ to $\square$ with a right adjoint. Then Watts' theorem implies that there is a bimodule X and a natural isomorphism $X \otimes M \to FM$. Because F is symmetric monoidal, we have a natural isomorphism

$$m_{M,N}\colon FM\square FN \to F(M \wedge N).$$

This translates to a natural isomorphism

$$m_{M,N}\colon \Lambda'_{Y,X} \otimes (X \otimes M) \otimes (Y \otimes N) \to X \otimes (\Lambda_{N,M} \otimes M \otimes N),$$

where Y is just another name for X. Taking $M = N = R$ gives us the desired isomorphism m. The unit isomorphism $\eta\colon K' \to FK$ is just the map $\eta\colon K' \to X \otimes K$. On the other hand, given X, m, and η, we define $FM = X \otimes M$, η in the obvious way, and $m_{M,N}$ by naturality from m. We leave to the reader the translation between the compatibility diagrams of F and the diagrams in the theorem. $\qquad\square$

3. Examples

In this section, we consider some examples of additive closed symmetric monoidal structures on R-modules. In particular, we find rings R where there are no such structures, where there is eactly one (up to additive symmetric monoidal equivalence), where there are exactly seven, and where there are a proper class.

The most obvious case is when R is a commutative ring, where $- \wedge -$ is the usual tensor product. This corresponds to $\Lambda_{B,A} = R$ with $x \odot_1 r = x \odot_2 r = xr = rx$. The maps a and c are both identity maps, the unit K is R itself, and the map ℓ is multiplication.

Now suppose R is a cocommutative Hopf algebra over a field K, with diagonal Δ, counit ϵ, and antipode χ. Let us write $\Delta x = \sum x' \otimes x''$, implicitly choosing a basis for R. The category of R-modules then becomes a closed symmetric monoidal category under the functor $M \wedge N$, where

$$M \wedge N = M \otimes_K N$$

with R-action defined by

$$x(m \otimes n) = \sum x'm \otimes x''n.$$

The unit is K, made into an R-module via ϵ. The closed structure is given by $K\text{-mod}(M, N)$, made into an R-module by the formula

$$(xf)(m) = \sum x''f((\chi(x'))m).$$

In case R is the group algebra $K[G]$ of a group, this is just the usual tensor product of group representations and its associated closed structure, with unit the trivial one-dimensional representation.

This closed symmetric monoidal structure corresponds to $\Lambda_{B,A} = R_B \otimes_K R_A$. The right R-module structures are just right multiplication on the two factors, and the left R-module structure is given by the same formula

$$x(m \otimes n) = \sum x'm \otimes x''n.$$

used in defining $M \wedge N$. The commutativity isomorphism is just the twist map $c \colon R_B \otimes_K R_A \to R_A \otimes_K R_B$, so $c(m \otimes n) = n \otimes m$. For this to be a map of left R-modules, we need R to be cocommutative. The unit isomorphism is the obvious isomorphism

$$\ell \colon (R_B \otimes_K R_K) \otimes_R K \cong R_B \otimes_K K \cong R_B,$$

so that $\ell(m \otimes n) = m\epsilon(n)$.

The associativity isomorphism

$$a \colon (R_C \otimes_K R_R) \otimes_R (R_B \otimes_K R_A) \to (R_R \otimes_K R_A) \otimes_R (R_C \otimes_K R_B)$$

is more confusing. One might guess that

$$a(x \otimes y \otimes z \otimes w) = y \otimes w \otimes x \otimes z$$

but is WRONG, because this map does not descend to the tensor product over R. Indeed, we must have

$$a(x \otimes y \otimes z \otimes w) = a(x \otimes 1 \otimes y(z \otimes w))$$
$$= \sum a(x \otimes 1 \otimes y'z \otimes y''w),$$

where $\Delta y = \sum y' \otimes y''$ as before. Now the guess

$$a(x \otimes 1 \otimes y'z \otimes y''w) = 1 \otimes y''w \otimes x \otimes y'z$$

is correct, and so our final formula for a is

$$a(x \otimes y \otimes z \otimes w) = \sum 1 \otimes y''w \otimes x \otimes y'z$$

Coassociativity of the diagonal on R then implies, after some painful checking, that this is a map of left R-modules. We leave to the excessively diligent reader the check that all the required coherence diagrams commute.

We would now like to classify all the additive closed symmetric monoidal structures on R-modules, up to additive symmetric monoidal equivalence, for various R. The easiest case is when the unit of the symmetric monoidal structure is R itself. This forces R to be commutative, and in this case there is only one such closed symmetric monoidal structure.

Proposition 3.1. *Suppose R is a ring equipped with an additive closed symmetric monoidal structure on the category of R-modules with unit isomorphic to R. Then R is commutative and this closed symmetric monoidal structure is additively symmetric monoidal equivalent to the usual one.*

We point out as a general rule that if the unit K' is isomorphic to some R-module K, we can always assume that the unit is K, up to symmetric monoidal equivalence. Indeed, we construct a new additive closed symmetric monoidal structure by leaving everything the same except the unit isomorphism ℓ, which we modify by the isomorphism so that the unit is K. The coherence diagrams still commute, so this is a closed symmetric monoidal structure. A symmetric monoidal equivalence between this new structure and the old one is given by the identity functor, with the unit map $K' \to K$ given by the isomorphism.

This proposition is saying that the coherence isomorphisms a, ℓ, and c of the standard symmetric monoidal structure are determined up to symmetric monoidal equivalence. In fact, the proof shows that a and c are exactly determined, though there is some room for flexibility in ℓ.

There is a quick proof that R must be commutative, since the endomorphism ring of the unit in a symmetric monoidal category must always be commutative, and if the unit is isomorphic to R that endomorphism ring is R as well. However, this also falls out of the coherence isomorphisms, so we re-prove this fact in the proof below.

Proof. The unit isomorphism shows that $\Lambda_{1,2} \cong R$ as a bimodule, using the first right module structure on Λ. We can then assume it is R using a symmetric monoidal equivalence. We want to show that $\Lambda_{1,2} = R$ as a 2-fold bimodule, not just as a bimodule. We can encode the second right module structure on Λ by defining $\sigma \colon R \to R$ by $\sigma(x) = 1 \odot_2 x$. Note that

$$z \odot_2 x = (z \cdot 1) \odot_2 x = z(1 \odot_2 x) = z\sigma(x),$$

so that σ gives us complete information on the second right module structure on $\Lambda_{1,2} = R$. We need to show that σ is the identity. A computation shows that σ is a ring homomorphism.

We first show that σ is an isomorphism. Consider the commutativity isomorphism $c \colon \Lambda_{A,B} \to \Lambda_{B,A}$. This has the property that

$$c(y) = c(1 \odot_1 y) = c(1) \odot_2 y = c(1)\sigma(y).$$

Since c is an isomorphism, we conclude that σ is an isomorphism.

We claim that associativity forces R to be commutative and σ to be the identity. Indeed, we have

$$a \colon \Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \to \Lambda_{\Lambda,A} \otimes \Lambda_{C,B}.$$

Both of these are isomorphic to R as left modules. In the domain, we have

$$x \otimes y = x(1 \otimes y) = x(\sigma(y) \otimes 1) = x\sigma(y)(1 \otimes 1),$$

and in the target we have

$$z \otimes w = z(1 \otimes w) = z(w \otimes 1) = zw(1 \otimes 1).$$

Thus a is determined by $a(1 \otimes 1)$. Since a is a left module isomorphism from (something isomorphic to) R to itself, $a(1 \otimes 1)$ must be a unit in R, which we can write as
$$a(1 \otimes 1) = \gamma \otimes 1$$
for some unit $\gamma \in R$. We will then have

$$a(x \otimes y) = x\sigma(y)(\gamma \otimes 1).$$

We first show that R is commutative. Choose arbitrary $r, s \in R$. Find y such that $\sigma(y) = s\gamma^{-1}$, using the fact that σ is an isomorphism. Then we have

$$a((1 \odot_C r) \otimes y) = a(1 \otimes y) \odot_C r.$$

But we have

$$a((1 \odot_C r) \otimes y) = a(r \otimes y) = r\sigma(y)\gamma(1 \otimes 1) = rs(1 \otimes 1),$$

and

$$a(1 \otimes y) \odot_C r = \sigma(y)\gamma(1 \otimes r) = s(r \otimes 1) = sr(1 \otimes 1).$$

12

We conclude that $rs = sr$, so R is commutative.

We must also have

$$a(1 \otimes (1 \odot_A z)) = a(1 \otimes 1) \odot_A z.$$

This means that

$$a(1 \otimes \sigma(z)) = (\gamma \odot_A z) \otimes 1$$

so

$$\sigma^2(z)\gamma(1 \otimes 1) = \gamma\sigma(z)(1 \otimes 1).$$

Thus $\sigma^2(z) = \sigma(z)$ for all z. Since σ is necessarily one-to-one, we conclude that $\sigma(z) = z$.

We now know that $\Lambda_{1,2}$ is isomorphic to R with both right module structures, and the left module structure, equal to the canonical one. Then the associativity isomorphism a of Theorem 2.3 is just an isomorphism of left R-modules from R to itself, so must be right multiplication by some unit r. But then the associativity pentagon shows that $r^2 = r^3$, so $r = 1$. Similarly, the commutativity isomorphism is right multiplication by a unit s, and, since we now know a is the identity, the commutativity-associativity hexagon says that $s^2 = s$, so $s = 1$. Finally, ℓ must also be multiplication by some unit t, but the coherence diagrams will commute no matter what t is. However, we can define a symmetric monoidal equivalence from the usual symmetric monoidal structure to the one with $\ell = t$ by letting F be the identity functor, letting the natural isomorphism m be the identity, and letting $\eta\colon R \to R$ be right multiplication by t. □

There are some simple cases where R is the only possible unit of a closed symmetric monoidal structure on the category of R-modules.

Theorem 3.2. *Let n be an integer. There is a unique additive closed symmetric monoidal category structure on the category of $\mathbb{Z}/n\mathbb{Z}$-modules, up to symmetric monoidal equivalence.*

This theorem was proved in case $n = 0$ by Foltz, Lair, and Kelly [FLK80]. We will generalize that case to principal ideal domains in Theorem 4.9.

Proof. A right or left $\mathbb{Z}/n\mathbb{Z}$-module structure on an abelian group is unique; we must have $nx = xn = x + x + \cdots + x$ for $n \geq 0$ and the negative of this for $n < 0$. Thus the 2-fold bimodule Λ needed to define a closed symmetric monoidal structure on $\mathbb{Z}/n\mathbb{Z}$-modules is simply a $\mathbb{Z}/n\mathbb{Z}$-module, with all of the module structures being the same. The unit isomorphism guarantees that Λ is in the Picard group of $\mathbb{Z}/n\mathbb{Z}$, which is trivial (see [Lam99, Example 2.22D]). Hence there is an isomorphism $f\colon \Lambda \to \mathbb{Z}/n\mathbb{Z}$. Proposition 3.1 completes the proof. □

The other simple case is when R is a division ring.

Theorem 3.3. *Suppose k is a division ring. If k is not a field, then there is no additive closed symmetric monoidal structure on the category of k-modules. If k is a field, there is a unique additive closed symmetric monoidal category structure on the category of k-modules, up to symmetric monoidal equivalence.*

Proof. Suppose we have a closed symmetric monoidal structure induced by $\Lambda_{B,A}$. The unit isomorphism

$$\Lambda_{B,K} \otimes_k K \cong k$$

shows that K has to be a one-dimensional vector space, so is isomorphic to k. Proposition 3.1 completes the proof. $\square$

Since the axioms for an additive closed symmetric monoidal structure on the category of R-modules do not actually mention R itself, the existence and number of such structures are both Morita invariant. Hence we get the following corollary.

Corollary 3.4. *Suppose R is a simple artinian ring, so that $R \cong M_n(D)$ for some division ring D and some integer n. If D is commutative, there is a unique additive closed symmetric monoidal structure on the category of R-modules, up to symmetric monoidal equivalence. If D is not commutative, then there is no additive closed symmetric monoidal structure on the category of R-modules.*

The unit of the closed symmetric monoidal structure on $M_n(k)$-modules, for k a field, is the unique simple left $M_n(k)$-module k^n.

To find a case where the additive closed symmetric monoidal structure is not unique, we consider the group ring $k[\mathbb{Z}/2]$. Even in this simple case, the classification of additive closed symmetric monoidal structures is quite involved, and will take the rest of this section and many lemmas. The ring $k[\mathbb{Z}/2]$ is both a commutative ring and a Hopf algebra, so we know there are at least two closed symmetric monoidal structures. The behavior of this group ring depends on whether the characteristic of k is 2, so we begin with this case.

We start by identifying the Hopf algebra structures on $k[\mathbb{Z}/2]$.

Lemma 3.5. *Suppose k is a field of characteristic 2, and $R = k[\mathbb{Z}/2] \cong k[x]/(x^2)$. There are two different isomorphism classes of Hopf algebra structures on k, one represented by H_0, in which $\Delta(x) = 1 \otimes x + x \otimes 1$, and one represented by H_1, in which $\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x$.*

We only use the associativity and unit axioms to prove this lemma, so it follows that every bialgebra structure on $k[\mathbb{Z}/2]$ is a cocommutative Hopf algebra structure.

Proof. Suppose we have a Hopf algebra structure on R. Then the counit ϵ must have $\epsilon(1) = 1$ since it is a k-algebra map, and $\epsilon(x) = 0$ since x is nilpotent. We must have

$$\Delta(x) = a_1(1 \otimes 1) + a_2(1 \otimes x) + a_3(x \otimes 1) + a_4(x \otimes x)$$

14

for some $a_1, a_2, a_3, a_4 \in k$. The fact that

$$0 = \Delta(x^2) = \Delta(x)^2$$

implies that $a_1 = 0$. The fact that Δ is counital implies that $a_2 = a_3 = 1$.

For $\alpha \in k$, let R_α denote R together with $\epsilon \colon R \to k$ given by $\epsilon(1) = 1$ and $\epsilon(x) = 0$ and $\Delta \colon R \to R \otimes_k R$ given by

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \alpha(x \otimes x)$$

A check shows this is a cocommutative (counital coassociative) bialgebra, and the preceding paragraph shows that every Hopf algebra structure on R is one of the R_α as a bialgebra. A Hopf algebra structure also has an antipode $\chi \colon R \to R$, but a check shows that the only possible antipode on R_α is given by the identity, and that this is indeed an antipode. Thus every Hopf algebra structure on R is one of the R_α.

Any isomorphism $f \colon R_\alpha \to R_\beta$ of Hopf algebras must be compatible with the counit, from which we conclude that $f(x) = rx$ for some nonzero $r \in k$. But then compatibility with Δ will hold if and only if $r\alpha = \beta$. So if β and α are both nonzero, $r = \alpha/\beta$ will yield the desired isomorphism, but R_0 is not isomorphic to any other R_α. $\qquad\square$

In both of these two Hopf algebra structures on $R = k[\mathbb{Z}/2]$ (where k has characteristic 2), the corresponding symmetric monoidal structure has $\Lambda = R \otimes_k R$, freely generated as a left R-module by $m = 1 \otimes 1$ and $m \odot_2 x = 1 \otimes x$. We also have $m \odot_1 x = x \otimes 1$. However, in H_0 we have

$$xm = 1 \otimes x + x \otimes 1 \quad \text{so} \quad m \odot_1 x = xm + m \odot_2 x.$$

In H_1, though, we have

$$xm = 1 \otimes x + x \otimes 1 + x \otimes x \quad \text{so} \quad m \odot_1 x = xm + (1 + x)m \odot_2 x.$$

In both cases, the unit isomorphism $\ell \colon \Lambda \otimes k \to R$ has $\ell(m \otimes 1) = 1$ and $\ell((m \odot_2 x) \otimes 1) = 0$. Also the commutativity isomorphism is defined by $c(m) = m$ (and thus $c(m \odot_2 x) = m \odot_1 x$). The associativity isomorphism has $a(m \otimes m) = m \otimes m$ in both cases, but in H_0 we have

$$a((m \odot_2 x) \otimes m) = 1 \otimes x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x = (m \odot_2 x) \otimes m + m \otimes (m \odot_2 x),$$

whereas in H_1 we have

$$\begin{aligned} a((m \odot_2 x) \otimes m) &= 1 \otimes x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x + 1 \otimes x \otimes 1 \otimes x \\ &= (m \odot_2 x) \otimes m + m \otimes (m \odot_2 x) + (m \odot_2 x) \otimes (m \odot_2 x). \end{aligned}$$

Let us refer to these k-linear closed symmetric monoidal structures as $\wedge_{H_0}$ and $\wedge_{H_1}$. Note that $X \wedge Y = X \otimes_k Y$ as k-modules in either case, it is just the action of $\mathbb{Z}/2$ that differs.

Theorem 3.6. *Suppose k is a field of characteristic 2, and let $R = k[\mathbb{Z}/2] \cong k[x]/(x^2)$. Suppose $- \wedge -$ is a k-linear closed symmetric monoidal structure on the category of R-modules with unit K. Then one of the following must hold.*

1. *$K \cong R$ and $- \wedge -$ is k-linearly equivalent to $- \otimes -$.*
2. *$K \cong k$ and $- \wedge -$ is k-linearly equivalent to $- \wedge_{H_1} -$ as a monoidal functor, but not necessarily as a symmetric monoidal functor.*
3. *$K \cong k$ and $- \wedge -$ is k-linearly equivalent to $- \wedge_{H_0} -$ by a natural isomorphism compatible with the left unit isomorphism ℓ, but not necessarily as a monoidal functor.*

In addition, we have

1. *The isomorphism classes $(- \wedge_{H_1} -, \beta)$ of closed symmetric monoidal structures with underlying monoidal functor $- \wedge_{H_1} -$ are parametrized by elements $\beta \in k$, where $c(m) = m + \beta x(m \odot_2 x)$ in the symmetric monoidal structure corresponding to β.*
2. *The isomorphism classes $(- \wedge_{H_0} -, \gamma)$ of closed monoidal structures with underlying functor $- \wedge_{H_0} -$ and unit isomorphism ℓ are parametrized by elements*
$$\gamma \in \{0\} \cup k^\times/(k^\times)^3,$$
   where
$$a(m \otimes m) = m \otimes m + \gamma x(m \odot_2 x \otimes m \odot_2 x)$$
   in the monoidal structure corresponding to γ.
3. *The isomorphism classes $(- \wedge_{H_0} -, \gamma, \beta)$ of closed symmetric monoidal structures with underlying monoidal functor $(- \wedge_{H_0} -, \gamma)$ are parametrized by elements β, where*
$$c(m) = m + \beta x(m \odot_2 x)$$
   in the symmetric monoidal structure corresponding to β, as follows.
   (a) *If $\gamma = 0$, then $\beta \in \{0\} \cup k^\times/(k^\times)^2$.*
   (b) *If $\gamma \neq 0$ and k does not have a primitive cube root of 1, then $\beta \in k$.*
   (c) *If $\gamma \neq 0$ and k does have a primitive cube root ω of 1, then $\beta = 0$ or a coset of the action of $\mathbb{Z}/3$ on $k^\times$ given by the action of ω.*

Just so we have a specific concrete example, this theorem says that when $k = \mathbb{Z}/2$, there are seven k-linear isomorphism classes of k-linear closed symmetric monoidal structures on $k[\mathbb{Z}/2]$-modules, one corresponding to the usual tensor product, two corresponding to different symmetric monoidal structures on $- \wedge_{H_1} -$, and four corresponding to different structures on the underlying unital functor $- \wedge_{H_0} -$.

We will prove this theorem through a series of lemmas.

Lemma 3.7. *Let $R = k[x]/(x^2)$ where k is a field, and suppose $- \wedge -$ is a k-linear closed symmetric monoidal structure on the category of R-modules, with unit K. Then either $K \cong R$ or $K \cong k$. If $K \cong R$, then $- \wedge -$ is equivalent to the usual tensor product.*

Proof. Any decomposition of K as a direct sum of R-modules induces a decomposition of R as a direct sum of R-bimodules, via the unit isomorphism $\Lambda_{R,K} \otimes K \cong R_R$. Since R is indecomposable, K must also be indecomposable. Take any nonzero $m \in K$. The annihilator of m is either (0) or (x). If it is (0), then K contains R as a submodule. But R is self-injective, so K would then contain R as a summand. Since K is indecomposable, $K \cong R$, and $- \wedge -$ is equivalent to the usual tensor product by Proposition 3.1. If this is not the case, then x annihilates all of K. But then K is a k-vector space, so since it is indecomposable, $K \cong k$. $\square$

Lemma 3.8. *Let $R = k[x]/(x^2)$ where k is a field, and suppose $- \wedge -$ is a k-linear closed symmetric monoidal structure on the category of R-modules, with unit k. Let Λ be the 2-fold bimodule inducing $- \wedge -$. Then there is an element $m \in \Lambda$ such that*

$$\Lambda = Rm \oplus R(m \odot_2 x) \cong R \oplus R(m \odot_2 x)$$

as left R-modules, where $\ell(m \otimes 1) = 1$ and $\ell((m \odot_2 x) \otimes 1) = 0$. Furthermore, $m \odot_2 x \neq 0$.

This lemma also says that Λ is a principal bimodule under the right action $\odot_2$, generated by m. We do not know yet whether Λ is a free bimodule on m, though we will prove this later.

Proof. Since $\Lambda_{B,k} \otimes k \cong R_B$ by the left unit isomorphism, we have

$$\Lambda/(\Lambda \odot_2 x) \cong R.$$

If we choose an $m \in \Lambda$ with $\ell(m \otimes 1) = 1$, then for any $\lambda \in \Lambda$, we have

$$\lambda = \ell(\lambda \otimes 1)m + n \odot_2 x = \ell(\lambda \otimes 1)m + (\ell(n \otimes 1)m + d \odot_2 x) \odot_2 x$$
$$= \ell(\lambda \otimes 1)m + \ell(n \otimes 1)(m \odot_2 x),$$

where n and d denote unknown elements of Λ. Thus Λ is generated as a left R-module by m and $m \odot_2 x$. Furthermore, if $rm \in Rm \cap R(m \odot_2 x)$, then

$$r = \ell(rm \otimes 1) = \ell(s(m \odot_2 x) \otimes 1) = 0,$$

so

$$\Lambda = Rm \oplus R(m \odot_2 x)$$

as left R-modules, as required.

Note also that
$$\lambda \odot_2 x = \ell(\lambda \otimes 1)(m \odot_2 x).$$

If $m \odot_2 x = 0$, this formula shows $\Lambda \odot_2 x = 0$. The commutativity isomorphism then implies $\Lambda \odot_1 x = 0$, so x acts trivially on the right on $\Lambda_{B,k} \otimes k \cong R_B$. Since this is false, we must have $m \odot_2 x \neq 0$. $\square$

At this point, we have not determined whether $m \odot_2 x$ generates a copy of k or a copy of R. This depends on whether $x(m \odot_2 x) = 0$ or not. Note, however, that $\ell((m \odot_1 x) \otimes 1) = x$, so we must have

$$m \odot_1 x = xm + b(m \odot_2 x).$$

for some $b \in R$. We will then have

$$(m \odot_2 x) \odot_1 x = (xm + b(m \odot_2 x)) \odot_2 x = xm \odot_2 x.$$

It will be helpful in what follows if we write $a = a^0 + a^1 x$ for elements $a \in R$, with $a^0, a^1 \in k$.

Lemma 3.9. *Suppose the characteristic of k is 2. In the situation of Lemma 3.8 and with the notation above, there is a commutativity involution $c\colon \Lambda_{A,B} \to \Lambda_{B,A}$ if and only if $b^0 = 1$, and in that case we have*

$$c(m) = (1 + \alpha x)m + (\beta x)(m \odot_2 x)$$

for some $\alpha, \beta \in k$, which can be anything. This implies

$$c(m \odot_2 x) = xm + (b + \alpha x)(m \odot_2 x).$$

Proof. Write

$$c(m) = rm + s(m \odot_2 x),$$

for some $r, s \in R$. Then

$$c(m \odot_2 x) = c(m) \odot_1 x = rxm + rb(m \odot_2 x) + sx(m \odot_2 x).$$

In order for c to have the desired properties, we need

$$c(m \odot_1 x) = c(m) \odot_2 x \quad \text{and} \quad c^2(m) = m.$$

In order for $c^2(m) = m$, computation shows that we need

$$r^2 + rsx = 1 \quad \text{and} \quad (rs + rsb + s^2 x)(m \odot_2 x) = 0.$$

The first equation is equivalent to $(r^0)^2 = 1$ and $2r^0 r^1 + r^0 s^0 = 0$, so $r^0 = \pm 1$ and $s^0 = -2r^1$. Since we are in characteristic 2, this means $r^0 = 1$ and $s^0 = 0$. We leave the second equation aside for the moment. In order for $c(m \odot_1 x) = c(m) \odot_2 x$, computation shows that we need

$$r(1 + b)x = 0 \quad \text{and} \quad (sx + rb^2 + sbx)(m \odot_2 x) = r(m \odot_2 x).$$

Since we know that $r^0 = \pm 1$, this first equation implies that $b^0 = -1$; since we are in characteristic 2, $b^0 = 1$. Computation then shows that, in characteristic 2, the conditions $r^0 = b^0 = 1$, $s^0 = 0$ guarantee that both equations

$$(rs + rsb + s^2 x)(m \odot_2 x) = 0 \quad \text{and} \quad (sx + rb^2 + sbx)(m \odot_2 x) = r(m \odot_2 x)$$

also hold, whether $x(m \odot_2 x) = 0$ or not. The proof is completed by taking $\alpha = r^1$ and $\beta = s^1$. $\qquad\square$

In fact, we can make $\alpha = 0$ by modifying our choice of m.

Lemma 3.10. *In the situation of Lemma* 3.9, *so in particular when the characteristic of k is* 2, *we can modify our choice of the bimodule generator m of Λ so that*

$$c(m) = m + \beta x(m \odot_2 x)$$

for some $\beta \in k$. In this case, $c(m \odot_2 x) = m \odot_1 x$.

From now on, we assume m is chosen as in Lemma 3.10.

Proof. Let $n = m + \alpha(m \odot_2 x)$. Then $\ell(n \otimes 1) = 1$, so n is a perfectly good bimodule generator for Λ. Note that

$$n \odot_2 x = m \odot_2 x \quad \text{and} \quad n \odot_1 x = xn + b(n \odot_2 x)$$

as before (after some calculation). However, we have

$$c(n) = n + \gamma x(n \odot_2 x),$$

after some calculation, for some $\gamma \in k$. □

We must now come to grips with the associativity isomorphism

$$a \colon \Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \to \Lambda_{\Lambda,A} \otimes \Lambda_{C,B}.$$

If Λ is a free bimodule generated by m, then both the domain and range of a are isomorphic to $R^{\oplus 4}$ as left R-modules, with summands generated by $m \otimes m$, $(m \odot_2 x) \otimes m$, $m \otimes (m \odot_2 x)$, and $(m \odot_2 x) \otimes (m \odot_2 x)$. If Λ has dimension 3, on the other hand, both the domain and range of a are isomorphic to $R \oplus k \oplus R$, with summands generated by $m \otimes m$, $(m \odot_2 x) \otimes m$ and $m \otimes (m \odot_2 x)$, respectively. Indeed, in this case $\Lambda \cong R \oplus k$ as a left R-module, so

$$\Lambda \otimes \Lambda \cong \Lambda \oplus \Lambda/(\Lambda \odot_\epsilon x) \cong R \oplus k \oplus R,$$

where $\epsilon = 2$ in the domain of a and $\epsilon = 1$ in the codomain of a.

Note that, whatever the dimension of Λ, in the domain of a we have

$$m \otimes x(m \odot_2 x) = (m \odot_2 x) \otimes (m \odot_2 x),$$

whereas in the codomain of a we have

$$m \otimes x(m \odot_2 x) = (m \odot_1 x) \otimes (m \odot_2 x) = xm \otimes (m \odot_2 x) + (1 + b^1 x)(m \odot_2 x) \otimes (m \odot_2 x).$$

In particular, if the dimension of Λ is 3, then

$$(m \odot_2 x) \otimes (m \odot_2 x) = 0$$

in the domain of a and

$$(m \odot_2 x) \otimes (m \odot_2 x) = xm \otimes (m \odot_2 x)$$

in the codomain of a.

With respect to this basis, write

$$a(m \otimes m) = (e_1, e_2, e_3, e_4) \quad \text{and} \quad a((m \odot_2 x) \otimes m) = (f_1, f_2, f_3, f_4).$$

If the dimension of Λ is 3, then we take $e_4 = f_4 = 0$. Note that if the dimension of Λ is 3, we must have

$$xa((m \odot_2 x) \otimes m) = 0,$$

so

$$f_1^0 = f_3^0 = 0.$$

Lemma 3.11. *With the above definitions, if a is a map of 3-fold bimodules, then the left and right unit coherence diagram commutes if and only if $e_1 = 1$, $e_2 = 0$, $f_1 = 0$, and $f_2 = 1$.*

Proof. Apply the coherence diagram to $m \otimes m \otimes 1$ and to $(m \odot_2 x) \otimes m \otimes 1$. $\quad\square$

We still have to determine what conditions are necessary for a to be a map of 3-fold bimodules.

Lemma 3.12. *In order for the map a defined above to be a map of 3-fold bimodules making the left and right unit coherence diagram commute, Λ must be the free bimodule on m, and*

$$e_1 = 1, \quad e_2 = 0, \quad e_3^0 = 0, \quad f_1 = 0, \quad f_2 = 1, \quad f_3 = 1, \quad f_4 = b^1 + e_3.$$

Proof. Of course, we know already that $e_1 = 1, e_2 = 0, f_1 = 0, f_2 = 1$, with $f_3^0 = 0$ if Λ has dimension 3. We have implicitly assumed that a is a map of left R-modules by defining it only in terms of generators. To ensure that a preserves the right module structure represented by A in

$$a \colon \Lambda_{C,\Lambda} \otimes \Lambda_{B,A} \to \Lambda_{\Lambda,A} \otimes \Lambda_{C,B},$$

we must have

$$a(m \otimes (m \odot_2 x)) = (1, 0, e_3, e_4) \odot_A x = (0, 1, 0, e_3)$$

and

$$a((m \odot_2 x) \otimes (m \odot_2 x)) = (0, 0, 0, f_3).$$

We now turn to the right module structure represented by B. Here we must have

$$a(m \otimes (m \odot_1 x)) = a(m \otimes m) \odot_B x.$$

Calculation shows that this forces $f_3 = 1$, and this rules out the case when the dimension of Λ is 3 (since $f_3^0 = 0$ if the dimension of Λ is 3). The same calculation shows that $f_4 = b^1 + e_3$. Further calculation shows that this is enough to ensure that a preserves the right module structure represented by B.

We now must ensure that a preserves the right module structure represented by C. More calculation of the relation

$$a((m \odot_1 x) \otimes m) = a(m \otimes m) \odot_C x$$

gives $e_3 x = 0$, so $e_3^0 = 0$. Further calculation implies that this is enough to make a preserve the right module structure represented by C. $\square$

We turn finally to the associativity pentagon.

Lemma 3.13. *Given that a satisfies the conditions of Lemma 3.12, a makes the associativity pentagon commute if and only if*

$$e_3 = 0, \quad f_4 = b^1, \quad e_4^0 = 0$$

and either

$$e_4^1 = 0 \quad or \quad b^1 = 0.$$

Proof. If we apply the associativity pentagon to $(m \odot_2 x) \otimes m \otimes m$, we eventually find

$$e_3 = 0 \quad \text{and} \quad e_4^0 = 0.$$

and so $f_4 = b^1 + e_3 = b^1$. Further computation with the associativity pentagon applied to $m \otimes m \otimes m$ eventually yields

$$e_4^1 b^1 = 0, \quad \text{so} \quad e_4^1 = 0 \quad \text{or} \quad b^1 = 0.$$

These conditions then make the associativity pentagon commute. $\square$

We are now left with determining which of these different Λ define additively equivalent symmetric monoidal structures. Recall from Theorem 2.4 that for Λ' and Λ to define additively equivalent symmetric monoidal structures, we need an element X of the bimodule Picard group, an isomorphism $\eta\colon k \to X \otimes k$, and an isomorphism

$$\Lambda'_{Y,X} \otimes X_A \otimes Y_B \to X \otimes \Lambda_{B,A}$$

making various diagrams commute, where $Y = X$.

Lemma 3.14. *Let $R = k[x]/x^2$. Up to isomorphism, the only invertible R-bimodules are the R_u, where u is a unit in k, $R_u = R$ as a left module, and $1 \odot x = ux$.*

This follows immediately from [Yek99, Lemma 3.3]. With all these lemmas in hand, we can now complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Let $X = R_u$, and suppose we have an isomorphism $\eta\colon k \to X \otimes k$ of left modules and an isomorphism

$$q\colon \Lambda'_{Y,X} \otimes X_A \otimes Y_B \to X \otimes \Lambda_{B,A}$$

of 2-fold bimodules making the compatibility diagrams of Theorem 2.4 commute. The map η is determined by $\eta(1) = 1 \otimes \rho$ for some nonzero ρ in k, and the map q is determined by

$$q(m \otimes 1 \otimes 1) = \sigma(1 \otimes m) + \tau(1 \otimes (m \odot_2 x))$$

with $\sigma, \tau \in R$, where we have used m for a bimodule generator in both Λ' and Λ satisfying the condition on $c(m)$ as in Lemma 3.10. Since q is a map of bimodules, it follows that

$$q((m \odot_2 x) \otimes 1 \otimes 1) = q(m \otimes x \otimes 1) = u^{-1}q(m \otimes (1 \odot x) \otimes 1) = u^{-1}\sigma(1 \otimes (m \odot_2 x)).$$

In particular, for q to be an isomorphism, we need σ to be a unit in R. We also need

$$q((m \odot_1 x) \otimes 1 \otimes 1) = u^{-1}q(m \otimes 1 \otimes (1 \cdot x)) = u^{-1}q(m \otimes 1 \otimes 1) \odot_1 x.$$

Further calculation with this last equation yields $b_\Lambda^1 = u^{-1}b_{\Lambda'}^1$.

This means that if $b_{\Lambda'}^1$ is nonzero, we can choose u so as to ensure $b_\Lambda^1 = 1$. Said another way, given Λ' with b^1 nonzero, we can define $\Lambda = R_{u^{-1}} \otimes \Lambda' \otimes R_u \otimes R_u$ for a suitable u and get an additively equivalent symmetric monoidal structure with $b^1 = 1$. Any isomorphism between a Λ' and a Λ both with $b^1 = 1$ must have $u = 1$, so we can think of it as an automorphism of Λ. It is still useful to use Λ' for the domain copy of Λ, because the choice of generator m as in Lemma 3.10 could be different in the two copies of Λ. We can then work through the compatibility diagrams of Theorem 2.4 in this case. The unit diagram forces $\sigma = \rho^1$, so $\sigma \in k$. In order for the commutativity diagram to commute, we need $\tau x = 0$, so $\tau^0 = 0$. Then one finds that the β in the commutativity isomorphism in both Λ' and Λ must be the same. Since we are in the case where $b^1 = 1$, the associativity isomorphism is completely determined by the preceding lemmas. Thus we find that if b^1 is nonzero, then the monoidal structure determined by Λ is additively equivalent to the one given by the Hopf algebra H_1. Since β does not change under these isomorphisms, it could be anything, so the different symmetric monoidal structures on $- \wedge_{H_1} -$ are classified by β.

Now suppose $b_{\Lambda'}^1 = b_\Lambda^1 = 0$. As above, we must have $\sigma = \rho^{-1}$ in order to ensure the unit compatibility diagram commutes. The commutativity compatibility diagram again forces $\tau^0 = 0$, but this time we have $\beta_\Lambda = u^{-2}\beta_{\Lambda'}$. Since $b^1 = 0$, the preceding lemmas allow for a nontrivial e_4^1 as well, so we must check the associativity compatibility diagram too. Painful computation then gives that the e_4^1 for Λ is u^{-3} times the e_4^1 for Λ'. Hence the different monoidal structures on $- \wedge_{H_0} -$ are classified by

$$\gamma \in \{0\} \cup k^\times/(k^\times)^3,$$

where γ is (the class of) e_4^1. If $\gamma = 0$, then we can use R_u to make the symmetric monoidal structure corresponding to β isomorphic to the one corresponding to $u^{-2}\beta$. Hence the symmetric monoidal structures when $\gamma = 0$ correspond to

$$\beta \in \{0\} \cup k^\times/(k^\times)^2.$$

22

When $\gamma \neq 0$, if we fix γ we can only use the R_u with $u^3 = 1$. If k has no primitive cube root of 1, then, the symmetric monoidal structures are parametrized by $\beta \in k$, but if k does have a primitive cube root of 1, the symmetric monoidal structures are parametrized by orbits of $\mathbb{Z}/3$ acting on k by multiplying by the primitive cube root of 1. $\qquad \square$

We now consider closed symmetric monoidal structures on $k[\mathbb{Z}/2]$-modules when the characteristic of k is not 2. In this case, there are two 1-dimensional representations: k_+, where $\mathbb{Z}/2$ acts trivially, and k_-, where $\mathbb{Z}/2$ acts by -1. The answer is now wildly different; there are a proper class of inequivalent closed symmetric monoidal structures!

Theorem 3.15. *Suppose k is a field whose characteristic is not 2, and let $R = k[\mathbb{Z}/2] \cong k[x]/(x^2 - 1)$. If $- \wedge -$ is a closed k-linear symmetric monoidal structure on the category of R-modules, then its unit K is isomorphic to R, k_+, or k_-. If the unit is isomorphic to R, then $- \wedge - \cong - \otimes -$ as k-linear symmetric monoidal functors. If $K \cong k_-$, then $- \wedge -$ is k-linearly symmetric monoidal equivalent to a closed k-linear symmetric monoidal structure whose unit is k_+. Given any R-module M, there is a k-linear symmetric monoidal structure for which the unit is k_+ and $k_- \wedge k_- = M$.*

It is easy to see that symmetric monoidal structures with nonisomorphic values of M cannot be equivalent. However, we do not know if there is more than one closed symmetric monoidal structure for a given M.

Proof. Let x denote the element $[1]$ of $k[\mathbb{Z}/2]$, so $R = k[\mathbb{Z}/2] \cong k[x]/(x^2 - 1)$. Since the characteristic is not 2, this ring is semisimple. Any module M splits as $M_+ \oplus M_-$, where M_+ is the 1-eigenspace of x and M_- is the -1-eigenspace of x. In particular, R itself so splits, with the splitting given by the orthogonal idempotents $e_+ = (1/2)(1+x)$ and $e_- = (1/2)(1-x)$. This produces a splitting

$$R\text{-mod} \cong k\text{-mod} \times k\text{-mod},$$

up to equivalence, of the entire category of R-modules. Thus every R-module is a direct sum of copies of k_+ and k_-, and there are no maps from k_+ to k_-, or from k_- to k_+. Note that there appears to be a notational clash here, as k_+ should denote the 1-eigenspace of k instead of what it actually denotes, which is k where $\mathbb{Z}/2$ acts as 1. However, there is not actually a clash, since the notation k by itself does not denote an R-module. The notation is so convenient we are willing to tolerate this apparent ambiguity.

This means that any 2-fold bimodule M will split as a direct sum of 8 2-fold bimodules $_aM_{b,c}$, where $a, b, c \in \{+, -\}$. Here $_aM_{b,c}$ is the sub 2-fold bimodule of M on which $\mathbb{Z}/2$ acts on the left according to a (trivially if $a = +$ and by -1 if a is $-$), $\mathbb{Z}/2$ acts on the right using the first action according to b, and $\mathbb{Z}/2$ acts on the right using the second action according to c. Of course, $_aM_{b,c}$ is a direct sum of copies of $_ak_{b,c}$, which denotes k with the left action of $\mathbb{Z}/2$ determined by a and the two right actions determined by b and c. The reader is warned that the same apparent notational clash arises here as in the previous

paragraph. The same comment applies to ordinary bimodules M, but now there will be 4 summands of the form $_aM_b$, and $_aM_b$ will be a direct sum of copies of $_ak_b$.

Given a k-linear closed symmetric monoidal structure, the corresponding 2-fold bimodule Λ splits as a direct sum of $_a\Lambda_{b,c}$, and the unit K splits as $K_+ \oplus K_-$. One can easily check that

$$_ak_{b,c} \otimes_2 k_c \cong k_{a,b},$$

where the subscript 2 on the tensor product indicates we are using the second right module structure to form the tensor product. On the other hand, if $d \neq c$, then

$$_ak_{b,c} \otimes_2 k_d = 0.$$

Now consider the left unit isomorphism of bimodules

$$\ell \colon\ _A\Lambda_{B,K} \otimes K \cong\ _AR_B.$$

Since the bimodule $R =\ _+k_+ \oplus\ _-k_-$, there must be a sign a such that

$$_+\Lambda_{+,a} =\ _+k_{+,a} \quad \text{and} \quad K_a = k_a$$

and a sign b such that

$$_-\Lambda_{-,b} =\ _-k_{-,b} \quad \text{and} \quad K_b = k_b.$$

If $a \neq b$, then $K = R$, and so Proposition 3.1 tells us that our closed symmetric monoidal structure is equivalent to $\otimes_R$. We can therefore assume $a = b$. The commutativity isomorphism tells us $_x\Lambda_{y,z}$ has the same dimension as $_x\Lambda_{z,y}$. Therefore, taking $z \neq a$, we find that $_z\Lambda_{a,z}$ is nonzero. If K_z were also nonzero, we would get a nonzero summand $_zk_a$ in the bimodule $\Lambda_{B,K} \otimes K \cong R$. Since we cannot have such a term, we conclude that $K = k_a$, so the unit is one-dimensional over k.

There is an obvious self-equivalence of the category of R-modules that permutes the two copies of k-mod. That is, it sends k_+ to k_-, and vice versa. Up to symmetric monoidal equivalence, then, we can assume the unit of our symmetric monoidal structure $-\wedge-$ is k_+. Let $M = k_- \wedge k_-$. Then, using the decomposition of $R =\ _+k_+ \oplus\ _-k_-$ as bimodules and the left unit isomorphism as above, we get

$$\Lambda \cong\ _+k_{+,+} \oplus\ _-k_{-,+} \oplus\ _-k_{+,-} \oplus M_{-,-},$$

where $M_{-,-}$ is the 2-fold bimodule whose underlying left module is M, and where x acts as -1 in both right module structures. Indeed, the term $_-\Lambda_{+,+}$, if nonzero, would produce a summand $_-k_+$ in R via the left unit, and the term $_+\Lambda_{-,+}$, if nonzero, would produce a summand $_+k_-$ in R. The commutativity isomorphism then tells us that the term $_+\Lambda_{+,-}$ must also be 0. This leaves us with the one-dimensional terms in Λ that we already understand, and the terms $_a\Lambda_{-,-}$. Computation using the definition

$$k_- \wedge k_- = \Lambda_{A,B} \otimes k_- \otimes k_-$$

24

tells us these terms together form $M_{-,-}$.

Now suppose we are given M. We want to construct a k-linear closed symmetric monoidal structure on R-modules with $k_- \wedge k_- = M$. We simply define $k_+ \wedge N = N \wedge k_+ = N$ for any R-module N, and $k_- \wedge k_- = M$. On morphisms, we note that there are no nonzero morphisms from k_+ to k_- or vice versa, and that every endomorphism of k_+ or k_- is given by multiplication by an element of k. So we define the induced morphism to be multiplication by the same element of k. Since every R-module is a direct sum of copies of k_+ and k_-, this defines $- \wedge -$ as a bifunctor. We define the left unit to be the identity. The commutativity isomorphism

$$c_{xy} \colon k_x \wedge k_y \to k_y \wedge k_x$$

is the identity, where x and y denote signs. We then extend through direct sums. The associativity isomorphism

$$a_{x,y,z} \colon (k_x \wedge k_y) \wedge k_z \to k_x \wedge (k_y \wedge k_z)$$

is the identity as well. We leave to the reader the check that the coherence diagrams hold. $\square$

4. Structural results

In the examples in the previous section, especially in Theorem 3.15, we saw that the 2-fold bimodule Λ can be very complicated. It does not have to be finitely generated, or even countably generated. However, it cannot be completely random either. Furthermore, in all the examples we have, the unit K of an additive symmetric monoidal structure on the category of R-modules is always a prinicipal R-module. It is tempting to wonder whether this always holds, or whether there are other properties that K must have.

In this section, we show that K is always a finitely generated module with commutative endomorphism ring, and that Λ is faithful in a very strong sense.

We begin by noting that tensoring with Λ reflects any property of morphisms that the tensor product preserves.

Proposition 4.1. *Suppose R is a ring and $\Lambda_{1,2}$ is a 2-fold R-bimodule that determines a closed symmetric monoidal structure on the category of R-modules with unit K. Let $\mathcal{P}$ be a replete class of morphisms of abelian groups such that:*

1. *If f is a morphism of left R-modules whose underlying morphism is in $\mathcal{P}$, then $A \otimes f$ is also in $\mathcal{P}$ for any right R-module A; and*
2. *If f is a morphism of right R-modules whose underlying morphism is in $\mathcal{P}$, then $f \otimes B$ is also in $\mathcal{P}$ for any left R-module B.*

Then, if f is a morphism of right R-modules, then $f \in \mathcal{P}$ if and only if $f \otimes \Lambda \in \mathcal{P}$. Similarly, if g is a morphism of left R-modules, then $g \in \mathcal{P}$ if and only if $\Lambda \otimes g \in \mathcal{P}$. This statement holds with either right module structure on Λ.

Recall that a class of morphisms is **replete** whenever $f \in \mathcal{P}$ and $f \cong g$ in the category of morphisms, then $g \in \mathcal{P}$. Said another way, if we have a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\
A' & \xrightarrow[\;g\;]{} & B'
\end{array}
$$

where i, j are isomorphisms, then $f \in \mathcal{P}$ if and only if $g \in \mathcal{P}$.

Proof. Suppose g is a morphism of left R-modules. By definition, if $g \in \mathcal{P}$, then $\Lambda \otimes g \in \mathcal{P}$. Conversely, suppose $\Lambda \otimes g \in \mathcal{P}$, where we use the leftmost right module structure on Λ. Then

$$g \cong R_R \otimes g \cong (\Lambda_{B,K} \otimes K) \otimes g \cong (\Lambda_{B,K} \otimes g) \otimes K,$$

and this is in $\mathcal{P}$ since $\Lambda_{B,K} \otimes g$ is so. We use the commutativity isomorphism to prove the same thing for the other right module structure on Λ. If f is a morphism of right R-modules, the proof is even easier. Indeed, we have

$$f \cong f \otimes R \cong f \otimes (\Lambda_{B,K} \otimes K) \cong (f \otimes \Lambda_{B,K}) \otimes K,$$

so if $f \otimes \Lambda \in \mathcal{P}$, then $f \in \mathcal{P}$. $\qquad\square$

Taking the class $\mathcal{P}$ to be the collection of zero morphisms, the collection of isomorphisms, and the collection of surjective maps gives us the following corollary.

Corollary 4.2. *Suppose R is a ring and $\Lambda_{1,2}$ is a 2-fold R-bimodule that determines a closed symmetric monoidal structure on the category of R-modules with unit K. Let f denote a morphism of right R-modules and let g denote a morphism of left R-modules.*

1. *$f \otimes \Lambda = 0$ if and only if $f = 0$. With either right module structure, $\Lambda \otimes g = 0$ if and only if $g = 0$.*
2. *$f \otimes \Lambda$ is an isomorphism if and only if f is so. With either right module structure, $\Lambda \otimes g$ is an isomorphism if and only if g is so.*
3. *$f \otimes \Lambda$ is a surjection if and only if f is so. With either right module structure, $\Lambda \otimes g$ is a surjection if and only if g is so.*

In particular, these imply that Λ is faithful.

Corollary 4.3. *Suppose R is a ring and $\Lambda_{1,2}$ is a 2-fold R-bimodule that determines a closed symmetric monoidal structure on the category of R-modules. Then Λ is faithful as a left or right R-module, with either right module structure.*

Proof. Choose $r \neq 0 \in R$. Then the map $R \to R$ that is left multiplication by r induces left multiplication by r on $\Lambda = R \otimes \Lambda$. Since $r \neq 0$, this map is also nonzero by Corollary 4.2. Hence r does not annihilate Λ, so Λ is faithful. Use right multiplication by r to see that Λ is faithful as a right R-module. $\qquad\square$

They also imply that K is finitely generated.

Theorem 4.4. *The unit K in an additive closed symmetric monoidal structure on the category of R-modules is finitely generated.*

Proof. In the unit isomorphism $\Lambda_{B,K} \otimes K \cong R_B$, write 1 as the image of a finite sum $\sum \lambda_i \otimes k_i$. Let K' denote the submodule of K generated by the k_i, and $j \colon K' \to K$ denote the inclusion. Then $\Lambda \otimes j$ is surjective, so j must also be. $\qquad\square$

We suspect that the unit K must in fact be a principal R-module, but we do not know how to prove this.

Another essential property of K, or the unit of any symmetric monoidal category, is that its endomorphisms commute with each other. Somewhat more is true in our case.

Theorem 4.5. *Suppose K is the unit of an additive closed symmetric monoidal structure on the category of R-modules. Then $\mathrm{End}_R(K)$ is a subring of the center $Z(R)$ of R.*

Proof. Suppose $f \in \mathrm{End}_R(K)$. Then $\Lambda \otimes f$ is a bimodule endomorphism of R, through the unit isomorphism. Any bimodule endomorphism of R must be given by $x \mapsto rx$ for some $r \in Z(R)$. This defines a ring homomorphism $\mathrm{End}_R(K) \to Z(R)$. If f is in the kernel of this homomorphism, then $\Lambda \otimes f = 0$, but then $f = 0$ by Corollary 4.2. $\qquad\square$

We note that $\mathrm{End}_R(K)$ can be a proper submodule of $Z(R)$, as for example when $R = k[x]/(x^2)$ and the unit is k (of characteristic 2).

It is pretty rare for an R-module to have a commutative endomorphism ring. Using the work of Vasconcelos [Vas70], for example, we can deduce the following corollary.

Corollary 4.6. *Suppose R is a commutative Noetherian ring with no nonzero nilpotent elements, and K is the unit of an additive closed symmetric monoidal structure on the category of R-modules. Then*

$$K \cong \mathfrak{a}/\mathfrak{b}$$

for some radical ideal $\mathfrak{b}$ and some ideal $\mathfrak{a} \supseteq \mathfrak{b}$ with the ideal quotient $(\mathfrak{b}{:}\mathfrak{a}) = \mathfrak{b}$.

Recall that the ideal quotient $(\mathfrak{b}{:}\mathfrak{a})$ is the set of all x such that $x\mathfrak{a} \subseteq \mathfrak{b}$.

Proof. Let $\mathfrak{b} = \mathrm{ann}(K)$, the annihilator of K. We have an obvious monomorphism of rings $R/\mathfrak{b} \to \mathrm{End}_R(K)$ that takes r to multiplication by r, But $\mathrm{End}_R(K)$ is a subring of R by Theorem 4.5. Hence $R/\mathfrak{b}$ is a subring of R, and therefore has no nilpotents, so $\mathfrak{b}$ is a radical ideal. Thus K is a finitely generated (by Theorem 4.4), faithful $R/\mathfrak{b}$-module with commutative endomorphism ring, and $R/\mathfrak{b}$ is a commutative Noetherian ring with no nonzero nilpotent elements. Vasconcelos [Vas70] proves in this situation that K is an ideal in $R/\mathfrak{b}$. Hence $K \cong \mathfrak{a}/\mathfrak{b}$ for some ideal $\mathfrak{a}$ of R. The condition on the ideal quotient is so that the annihilator of K will in fact be $\mathfrak{b}$. $\qquad\square$

Note that, if M is a submodule of the unit K, then the image of $\Lambda \otimes M$ in $\Lambda \otimes K \cong R$ will be a sub-bimodule of R, and hence a two-sided ideal.

Corollary 4.7. *Suppose K is the unit of an additive closed symmetric monoidal structure on the category of R-modules. Then every nonzero proper submodule of K gives rise to a nonzero proper two-sided ideal of R. Hence, if R is a simple ring, then K is a simple left module.*

We note that a simple commutative ring is of course a field, but there are many simple noncommutative rings that are not division rings. We would like to be able to say that the map from nonzero proper submodules of K to two-sided ideals of R is one-to-one, but we do not know if this is true.

Proof. Suppose L is a proper submodule of K. Then the maps $L \to K$ and $K \to K/L$ are both nonzero, so they remain so after tensoring with $\Lambda_{B,K}$ by Corollary 4.2. Hence the image of $\Lambda_{B,L} \otimes L$ is a nonzero proper subbimodule of $\Lambda_{B,K} \otimes K = R_B$. $\square$

As above, we do not know if this map from submodules of K to two-sided ideals in R is one-to-one, but it is on direct summands of K.

Corollary 4.8. *Suppose K is the unit of an additive closed symmetric monoidal structure on the category of R-modules. There is a one-to-one map from isomorphism classes of direct summands of K to central idempotents in R. In particular, if R is indecomposable as a ring, then K is indecomposable as an R-module.*

Proof. Suppose M is a direct summand of K, so that there is a retraction $f \colon K \to M$. Tensoring with Λ gives us a retraction of bimodules $R \to \Lambda \otimes M$. The composite

$$R \to \Lambda \otimes M \to R$$

must be multiplication by a central idempotent e of R, with $\Lambda \otimes M = eR$. The bimodule $\Lambda \otimes M$ determines e [Lam01, Exercise 22.2], and we can recover M from $\Lambda \otimes M$, up to isomorphism, by tensoring witk K. $\square$

Finally, we put these structural results together to show that there is only one closed symmetric monoidal structure on R-modules when R is a principal ideal domain that does not contain a field.

Theorem 4.9. *Suppose R is a principal ideal domain with an additive closed symmetric monoidal structure on the category of R-modules. Then the unit K is either isomorphic to R or to a field R/p for some prime element of R. Thus, if R does not contain a field, there is a unique additive closed symmetric monoidal structure on the category of R-modules, up to symmetric monoidal equivalence.*

Note that, if k is a field, then $k[x]$ is a principal ideal domain and can be made into a Hopf algebra over k with $\Delta(x) = 1 \otimes x + x \otimes 1$, so there are at least two inequivalent additive closed symmetric monoidal structures on $k[x]$-modules.

Proof. We know from Theorem 4.4 and Corollary 4.8 that the unit K of a closed symmetric monoidal structure on R-modules is an indecomposable finitely generated R-module. Since R is a principal ideal domain, this means $K \cong R/p^n$ for some prime element $p \in R$. On the other hand, we know from Corollary 4.6 that the annihilator $\mathfrak{b}$ of K must be a radical ideal, so $n = 1$ and $K \cong R/p$. According to Theorem 4.5, $\operatorname{End}_R(R/p) = R/p$ must be a subring of R. Thus if R does not contain a field, we must have $p = 0$, so $K \cong R$. Then Proposition 3.1 completes the proof. $\qquad\square$

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