SMITH IDEALS OF STRUCTURED RING SPECTRA

MARK HOVEY

ABSTRACT. Pursuing ideas of Jeff Smith, we develop a homotopy theory of ideals of monoids in a symmetric monoidal model category. This includes Smith ideals of structured ring spectra and of differential graded algebras. Such Smith ideals are NOT subobjects, and as a result the theory seems to require us to consider all Smith ideals of all monoids simultaneously, rather then restricting to the Smith ideals of one particular monoid. However, we can take a quotient by a Smith ideal and get a monoid homomorphism. In the stable case, we show that this construction is part of a Quillen equivalence between a model category of Smith ideals and a model category of monoid homomorphisms.

INTRODUCTION

There has been a great deal of interest in the last several decades in the theory of structured ring spectra in algebraic topology. In simple terms, these are cohomology theories with cup products that are infinitely homotopy associative, though we now think of them just as monoids in a suitable model category of spectra. In trying to develop the ring theory of these ring spectra, one runs into a basic problem right away: a ring spectrum R has no elements. Thus even a simple concept like an ideal of a ring spectrum has not been developed. Of course, a ring spectrum R has associated to it its homotopy ring $\pi_* R$, and we can talk of ideals here. But right away we run into trouble. For $R = S$, the sphere spectrum, and the element $2 \in \pi_0 S$, the cofiber of the times 2 map, the mod 2 Moore spectrum, is not a ring spectrum even up to homotopy. So (2) cannot be an ideal of S, though it is an ideal of $\pi_* S$.

In a talk given on June 6, 2006, Jeff Smith outlined a theory of ideals of ring spectra. He did much more than this, using his theory to discuss algebraic K-theory of pushouts. Bob Bruner kindly sent me his notes from the talk, and some of his thoughts with Dan Isaksen [BI07] about Smith's work. This paper is my interpretation of some of that work. I stress that the material in this paper was essentially the background material in Smith's talk, so there is much more left to be done.

Obviously we cannot think of an ideal in a ring spectrum as a collection of elements. We should instead think of it as a map $f\colon I \to R$. If I is a two-sided ideal, this should be a monomorphism of R-bimodules. However, we know that "monomorphism" is not a good homotopy-theoretic concept, because every map should be homotopic to a monomorphism. So a Smith ideal should be a map $f\colon I \to R$ of R-bimodules that is trying to be a monomorphism in some way.

We could continue in this vein, but let us instead think for a moment about the category $\operatorname{Arr} \mathcal{C}$ of maps in a symmetric monoidal category $\mathcal{C}$. A Smith ideal is

Date: September 30, 2013.

going to be a certain kind of object in $\operatorname{Arr}\mathcal{C}$, and since it is related to monoids in $\mathcal{C}$, maybe a Smith ideal should simply be a monoid in $\operatorname{Arr}\mathcal{C}$. For this, we need a symmetric monoidal structure on $\operatorname{Arr}\mathcal{C}$ induced by the one from $\mathcal{C}$. There is one obvious one, the **tensor product monoidal structure**, in which the monoidal product of $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ is

$$f \otimes g\colon X_0 \otimes Y_0 \to X_1 \otimes Y_1.$$

However, a monoid for this monoidal structure is just a monoid homomorphism, so this cannot be the right monoidal structure.

The correct monoidal structure on $\operatorname{Arr}\mathcal{C}$ is the **pushout product monoidal structure**, in which the monoidal product $f \mathbin{\Box} g$ of f and g is the map

$$f \mathbin{\Box} g\colon (X_0 \otimes Y_1) \amalg_{(X_0 \otimes Y_0)} (X_1 \otimes Y_0) \to X_1 \otimes Y_1.$$

This is the same pushout product used in the definition of a symmetric monoidal model structure. A monoid for this monoidal structure turns out to be a monoid R, an R-bimodule I, and a map of R-bimodules $j\colon I \to R$, such that

$$\mu(1 \otimes j) = \mu(j \otimes 1)\colon I \otimes I \to I,$$

where μ denotes both the right and left multiplication of R on I. This is a Smith ideal and is equivalent (by using work of Bruner and Isaksen [BI07]) to the definition Smith gave in his talk in terms of enriched functors.

In this paper, we first construct the pushout product and tensor product monoidal structures on $\operatorname{Arr}\mathcal{C}$ in Section 1, and show the perhaps suprising result that the cokernel is a symmetric monoidal functor from the pushout product monoidal structure to the tensor product monoidal structure. This means that the cokernel of a Smith ideal is a ring spectrum.

We then need to consider homotopy theory. In Section 3 and Section 2 we develop two model category structures on $\operatorname{Arr}\mathcal{C}$. In the projective model structure, a morphism in $\operatorname{Arr}\mathcal{C}$ is a fibration or weak equivalence if and only if its two components are so in $\mathcal{C}$. The projective model structure is compatible with the tensor product monoidal structure. In the injective model structure, a morphism is a cofibration or weak equivalence if and only if its two components are so in $\mathcal{C}$. The injective model structure is compatible with the pushout product monoidal structure. As a result, we get model categories of Smith ideals and monoid homomorphisms, and the cokernel is a left Quillen functor from Smith ideals to monoid homomorphisms. To take the homotopically meaningful quotient of a ring spectrum R by a Smith ideal $j\colon I \to R$, we must first take a cofibrant approximation $j'\colon I' \to R'$ in the model category of Smith ideals, and then take R'/I'. Note that R' is weakly equivalent as a monoid to R, but of course it is not R.

In Section 4 we prove that the cokernel is a Quillen **equivalence** in case our model category $\mathcal{C}$ is stable, as for example any model category of spectra. Thus, in the stable case, a Smith ideal is the same information homotopically as a monoid homomorphism. This is a bit disconcerting, since this says, for example, that every structured ring spectrum R is weakly equivalent to the quotient of the sphere S by some Smith ideal. This would be like saying that every ring is a quotient of the integers $\mathbb{Z}$. But we have to remember that we are free to add an enormous contractible ring spectrum to S and take a Smith ideal of that, and this is why we can get R as a quotient of S up to homotopy. There is also an appendix where we discuss some technical issues.

There is much work still to be done. For example, given any map $f\colon I \to R$, what is the Smith ideal generated by f? The answer must be the free monoid Tf on f in the pushout product monoidal structure. The problem is that this is a Smith ideal of the free monoid TR on R, not of R itself. The author has tried various ways to convert this to a Smith ideal of R, without success. It is possible that we have to accept that the ideal generated by $2\colon S \to S$ is an ideal of $TS = S[x]$, not of S. Even in that case, it would be extremely interesting to know the quotient of $S[x]$ by $T(2)$.

A related question is to fix the quotient issue. We have the wrong definition of quotient if every ring spectrum is a quotient of S. Our weakening of the definition of ideal means that we should strengthen the definition of quotient. One way to do it is to define a monoid homomophism $p\colon R \to S$ to be a **strong quotient** if the map

$$S \otimes_R QN \to N$$

is a weak equivalence for all fibrant S-modules N, where Q denotes cofibrant replacement in the category of R-modules. This would mean that the homotopy category of S-modules is a fully faithful subcategory of the homotopy category of R-modules. Is this a useful notion? Is it possible to use Smith ideals to help classify strong quotients of ring spectra? We don't know the answers.

Finally, we have not dealt with the commutative situation at all. It is generally much more difficult to determine whether commutative monoids in a symmetric monoidal model category inherit a model structure, as there are several well-known instances where it is false. But if they do, one might hope to define a commutative Smith ideal as a commutative monoid in the arrow category. This would give us a potentially different notion of ideal, and we would like to know how different.

Obiovusly, this talk owes everything to Jeff Smith's 2006 lecture, and the author thanks him for having such wonderful ideas. The author is certain that he has not come close to the depth of Smith's vision about the subject. The author also thanks Bob Bruner and Dan Isaksen for sharing their very helpful thoughts on Smith ideals.

Throughout this paper, $\mathcal{C}$ will denote a bicomplete closed symmetric monoidal category with monoidal product $A \otimes B$, closed structure $\operatorname{Hom}(A, B)$, and unit S. Usually $\mathcal{C}$ will also be a model category, and in that case we will always assume the model structure is compatible the monoidal structure, so that $\mathcal{C}$ is a symmetric monoidal model category. Facts about model categories can generally be found in [Hov99], among other places, but we try to cite specific facts more precisely.

1. The arrow category of a symmetric monoidal model category

In this section, we show that there are two different closed symmetric monoidal structures on the arrow category $\operatorname{Arr}\mathcal{C}$ of our closed symmetric monoidal category $\mathcal{C}$. We also discuss the cokernel functor, which is a symmetric monoidal functor from one of these structures to the other. Finally, we discuss monoids and modules in each of these symmetric monoidal structures. We postpone all discussion of homotopy theory to the next section.

Recall that an object $\operatorname{Arr}\mathcal{C}$ is simply a map f of $\mathcal{C}$. For notational convenience, we will often write f as $f\colon X_0 \to X_1$. A map $\alpha\colon f \to g$ in $\operatorname{Arr}\mathcal{C}$ is then a commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & Y_1 \\ {\scriptstyle\alpha_0}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_1} \\ Y_0 & \xrightarrow[g]{} & Y_1. \end{array}$$

Note that $\operatorname{Arr}\mathcal{C}$ is the category of functors from the category $\mathcal{J}$ with two objects 0 and 1 and one non-identity map $0 \to 1$ to $\mathcal{C}$. Hence, since $\mathcal{C}$ is bicomplete, so is $\operatorname{Arr}\mathcal{C}$, with limits and colimits taken objectwise. In addition, if $\mathcal{C}$ is locally presentable, so is $\operatorname{Arr}\mathcal{C}$, as a category of small diagrams in a locally presentable category [AR94].

There are obvious functors $\mathrm{Ev}_0, \mathrm{Ev}_1\colon \operatorname{Arr}\mathcal{C} \to \mathcal{C}$, and these have left and right adjoints. We leave the following lemma to the reader.

Lemma 1.1. *Suppose $\mathcal{C}$ is a closed symmetric monoidal category. The evaluation functors $\mathrm{Ev}_0, \mathrm{Ev}_1\colon \operatorname{Arr}\mathcal{C} \to \mathcal{C}$ have left adjoints L_0, L_1 and right adjoints U_0, U_1. We have*

$$L_0(X) = U_1 X = 1_X, L_1(X) = 0 \to X, \text{ and } U_0(X) = X \to *.$$

We now discuss monoidal structures on $\operatorname{Arr}\mathcal{C}$, of which there are at least two.

Theorem 1.2. *Let $\mathcal{C}$ be a closed symmetric monoical category. The category $\operatorname{Arr}\mathcal{C}$ has two different closed symmetric monoidal structures. In the **tensor product monoidal structure**, the monoidal product of $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ is given by*

$$f \otimes g\colon X_0 \otimes Y_0 \to X_1 \otimes Y_1.$$

The unit is $L_0 S$, and the closed structure is given by the projection map

$$\operatorname{Hom}_{\otimes}(f,g) = \operatorname{Hom}(X_0, Y_0) \times_{\operatorname{Hom}(X_0,Y_1)} \operatorname{Hom}(X_1, Y_1) \to \operatorname{Hom}(X_1, Y_1).$$

*In the **pushout product monoidal structure**, the monoidal product of f and g is the pushout product*

$$(X_0 \otimes Y_1) \amalg_{(X_0 \otimes Y_0)} (X_1 \otimes Y_0) \to X_1 \otimes Y_1.$$

The unit is $L_1 S$, and the closed structure $\operatorname{Hom}_{\square}(f,g)$ is given by

$$\operatorname{Hom}_{\square}(f,g) = \operatorname{Hom}(X_1, Y_0) \to \operatorname{Hom}(X_0, Y_0) \times_{\operatorname{Hom}(X_0,Y_1)} \operatorname{Hom}(X_1, Y_1).$$

We note that $\operatorname{Arr}\mathcal{C}$ itself, with either closed symmetric monoidal structure above, is an appropriate input for Theorem 1.2. That is, we can iterate the construction and get various closed symmetric monoidal structures on $\operatorname{Arr}\operatorname{Arr}\mathcal{C}$, the category of commutative squares in $\mathcal{C}$. We can of course continue this iteration.

Proof. We leave the majority of this proof to the reader. The most annoying thing to construct is the associativity isomorphism for the pushout product monoidal structure. Here the idea is to use the properties of colimits to conclude that both $(f \square g) \square h$ and $f \square (g \square h)$ are isomorphic to

$$\operatorname{colim}_{(i,j,k)\neq(1,1,1)}(X_i \otimes Y_j \otimes Z_k \to X_1 \otimes Y_1 \otimes Z_1,$$

where $h\colon Z_0 \to Z_1$. $\square$

It is useful to record how L_0 and L_1 interact with these symmetric monoidal structures.

Lemma 1.3. *Let $\mathcal{C}$ be a closed symmetric monoidal category. As a functor to the tensor product monoidal structure, $L_0\colon \mathcal{C} \to \operatorname{Arr}\mathcal{C}$ is symmetric monoidal, whereas $L_1X \otimes f = L_1(X \otimes \operatorname{Ev}_1 f)$. As a functor to the pushout product monoidal structure, L_1 is symmetric monoidal, whereas $L_0(X) \mathbin{\square} f = L_0(X \otimes \operatorname{Ev}_1 f)$.*

Again, we leave the proof to the reader.

Since we are interested in exploring the relationship between ideals and quotients, we need to understand the functor coker: $\operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}$. Here we need to assume $\mathcal{C}$ is pointed. The cokernel functor is defined by

$$\operatorname{coker}(f\colon A \to B) = (B \to \operatorname{coker} f),$$

where we rely on context to distinguish between $\operatorname{coker} f$ as a map in $\mathcal{C}$ and as an object in $\mathcal{C}$.

We can now see the importance of the two different symmetric monoidal structures on $\operatorname{Arr}\mathcal{C}$.

Theorem 1.4. *Suppose $\mathcal{C}$ is a pointed closed symmetric monoidal category. The functor* coker: $\operatorname{Arr}\mathcal{C} \to \operatorname{Arr}\mathcal{C}$ *is a (strongly)symmetric monoidal functor from the pushout product monoidal structure to the tensor product monoidal structure. Its right adjoint is the kernel.*

Proof. First note that the cokernel preserves the units, since

$$\operatorname{coker}(0 \to S) = (S \xrightarrow{=} S).$$

To see the cokernel is monoidal, we use the fact that pushouts commute with each other. More precisely, we have the following commutative diagram, for maps $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$.

$$\begin{tikzcd}
X_1 \otimes Y_1 & X_0 \otimes Y_1 \arrow[l] \arrow[r] & 0 \\
X_1 \otimes Y_0 \arrow[u] & X_0 \otimes Y_0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[u, equal] \arrow[d, equal] \\
X_1 \otimes Y_0 \arrow[u, equal] \arrow[r, equal] & X_1 \otimes Y_0 \arrow[r] & 0
\end{tikzcd}$$

If we take vertical pushouts in this diagram, we get the diagram

$$X_1 \otimes Y_1 \xleftarrow{f \square g} (X_0 \otimes Y_1) \amalg_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \to 0,$$

whose pushout is $\operatorname{coker}(f \square g)$. On the other hand, if we take the horizontal pushouts instead, we get the diagram

$$\operatorname{coker} f \otimes Y_1 \leftarrow \operatorname{coker} f \otimes Y_0 \to 0,$$

whose pushout is $\operatorname{coker} f \otimes \operatorname{coker} g$. Since pushouts commute with each other, we see that

$$\operatorname{coker}(f \mathbin{\square} g) \cong \operatorname{coker} f \otimes \operatorname{coker} g,$$

both as objects in $\mathcal{C}$ and as maps in $\mathcal{C}$. We leave the check that the required coherence diagrams commute to the reader, as also the proof that the kernel is right adjoint to the cokernel. $\square$

We now consider the monoids and modules in our two symmetric monoidal structures on $\operatorname{Arr}\mathcal{C}$.

Proposition 1.5. *Let $\mathcal{C}$ be a closed symmetric monoidal category. A monoid in the tensor product monoidal structure on* $\operatorname{Arr}\mathcal{C}$ *is simply a monoid homomorphism* $p\colon R_0 \to R_1$ *in* $\mathcal{C}$*. A module over the monoid* p *is an* R_0*-module* M_0*, an* R_1*-module* M_1*, and an* R_0*-module map* $f\colon M_0 \to M_1$*, where* M_1 *is an* R_0*-module by restricting scalars through* p*.*

Said another way, with the tensor product monoidal structure, the monoids in the arrow category of $\mathcal{C}$ are the arrows in the monoid category of $\mathcal{C}$.

Proof. A monoidal structure on p consists of a unit map

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d] & S \arrow[d] \\ R_0 \arrow[r, "p"'] & R_1 \end{tikzcd}$$

and a multiplication map

$$\begin{tikzcd} R_0 \otimes R_0 \arrow[r, "p\otimes p"] \arrow[d] & R_1 \otimes R_1 \arrow[d] \\ R_0 \arrow[r, "p"'] & R_1. \end{tikzcd}$$

These are just equivalent to a unit and a multiplication for R_0 and R_1 that are preserved by p. The associativity and unit diagrams says that the multiplications on R_0 and R_1 are associative and unital. The proof for modules is similar. $\square$

The monoids and modules in the pushout product monoidal structure are much more interesting.

Definition 1.6. Let $\mathcal{C}$ be a closed symmetric monoidal category. A **Smith ideal** in $\mathcal{C}$ is a monoid $j\colon I \to R$ in the pushout product monoidal structure on $\operatorname{Arr}\mathcal{C}$, which we often denote (R, I). Given a Smith ideal (R, I), a (right) (R, I)**-module** is a right module in the pushout product monoidal structure on $\operatorname{Arr}\mathcal{C}$ over the monoid j.

The "Smith" in Smith ideal is Jeff Smith. We will discuss the relationship between our definition and the definition given by Smith below.

Our first job is to unwind these definitions.

Proposition 1.7. *Let $\mathcal{C}$ be a closed symmetric monoidal category. A Smith ideal* $j\colon I \to R$ *in* $\mathcal{C}$ *is equivalent to a monoid* R *in* $\mathcal{C}$*, an* R*-bimodule* I *in* $\mathcal{C}$*, and a morphism* $j\colon I \to R$ *of* R*-bimodules such that the diagram below commutes.*

$$\begin{tikzcd} I \otimes_R I \arrow[r, "j\otimes 1"] \arrow[d, "i\otimes j"'] & R \otimes_R I \arrow[d, "\cong"] \\ I \otimes_R R \arrow[r, "\cong"'] & I \end{tikzcd}$$

Of course the tensor products in the commutative diagram above are tensor products of bimodules, so involve both the right and left action of R on I.

Proof. A monoid structure on j is given by a multiplication map $\mu\colon j \square j \to j$ and a unit $\eta\colon L_1S \to j$ making the usual associativity and unit diagrams commute. Writing $j\colon I \to R$, we see that η is equivalent to a map $\eta\colon S \to R$, and μ is equivalent to the commutative diagram below.

$$\begin{tikzcd}
(I \otimes R) \amalg_{I\otimes I} (R \otimes I) \arrow[r] \arrow[d] & R \otimes R \arrow[d] \\
I \arrow[r, "j"'] & R
\end{tikzcd}$$

Thus the existence of μ is equivalent to a multiplication on R, a left and right multiplication of R on I that agree on $I \otimes I$, and the fact that j preserves those multiplications. If we write out $j \square (j \square j)$ carefully, we see that its domain is an iterated pushout involving $I \otimes (R \otimes R)$, $R \otimes (I \otimes R)$, and $R \otimes (R \otimes I)$, and its codomain is $R \otimes (R \otimes R)$. Similarly, the domain of $(j \square j) \square j$ is an iterated pushout involving $(I \otimes R) \otimes R$, $(R \otimes I) \otimes R$, and $(R \otimes R) \otimes I$, and the codomain is $(R \otimes R) \otimes R$. Therefore, the associativity diagram for μ is equivalent to associativity of the left multiplication of R on I, the fact that the left and right multiplications of R and I commute with each other, associativity of the right multiplication of R on I, and the associativity of the multiplication on R. The unit diagrams are equivalent to η acting as a left and right unit on R and on I. □

It was pointed out by Bruner and Isaksen [BI07] that the data above, an R-bimodule map $j\colon I \to R$ making the diagram in Proposition 1.7 commute, are equivalent to a monoid R and a $\mathcal{C}$-category $\mathcal{I}$ with two objects a and b where $\mathcal{I}(a, a) = \mathcal{I}(b, b) = \mathcal{I}(a, b) = R$ with all compositions involving only these morphism objects being multiplication in R. This was the definition of an ideal of R given by Jeff Smith in his talk of June 6, 2006. The R-bimodule I is then $\mathcal{I}(b, a)$.

A map of Smith ideals $\alpha\colon (R, I) \to (R', I')$ is of course just a map of monoids in $\operatorname{Arr} \mathcal{C}$. This unwinds to a map of monoids $\alpha_1\colon R \to R'$ and a map of R-bimodules $\alpha_0\colon I \to I'$, where I' is an R-bimodule through restriction of scalars, such that $\alpha_1 j = j'\alpha_0$.

We also unwind the definition of a module over a Smith ideal.

Proposition 1.8. *If $j\colon I \to R$ is a Smith ideal in a closed symmetric monoidal category $\mathcal{C}$, an (R, I)-module is equivalent to maps of right R-modules $f\colon M_0 \to M_1$ and $\phi\colon M_1 \otimes_R I \to M_0$ such that the diagrams*

$$\begin{tikzcd}
M_0 \otimes_R I \arrow[r, "1\otimes j"] \arrow[d, "f\otimes 1"'] & M_0 \otimes_R R \arrow[d, "\cong"] \\
M_1 \otimes_R I \arrow[r, "\phi"'] & M_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
M_1 \otimes_R I \arrow[r, "1\otimes j"] \arrow[d, "\phi"'] & M_1 \otimes_R R \arrow[d, "\cong"] \\
M_0 \arrow[r, "f"'] & M_1
\end{tikzcd}$$

commute.

The result for left R-modules is similar; this time we have maps $f\colon M_0 \to M_1$ and $\phi\colon I \otimes_R M_1 \to M_0$ of left R-modules making analogous diagrams commute.

Proof. Let $f\colon M_0 \to M_1$ denote an object in $\operatorname{Arr}\mathcal{C}$. A right j-module structure on f is an action map $\mu\colon f \,\Box\, j \to f$ making associativity and unit diagrams commute. Since we have

$$f \,\Box\, j\colon (M_0 \otimes R) \amalg_{M_0 \otimes I} (M_1 \otimes I) \to M_1 \otimes R,$$

the map μ is equivalent to right multiplications μ_0, μ_1 of R on M_0 and M_1 that are preserved by f and a map $\phi_0\colon M_1 \otimes I \to M_0$ such that $f\phi_0 = \mu_1(1 \otimes j)$ and $\phi_0(f \otimes 1) = \mu_0(1 \otimes j)$. Careful consideration of the associativity diagram shows that μ is associative when μ_0 and μ_1 are associative, when ϕ_0 descends to $\phi\colon M_1 \otimes_R I \to M_0$, and when ϕ is a right R-module map. Of course, the unit diagram commutes exactly when the action of R on M_0 and M_1 is unital. □

A map of (R, I)-modules $M \to N$ is of course a pair of right R-module maps $M_0 \to N_0$ and $M_1 \to N_1$ making the evident diagrams involving f and ϕ commute.

Whenever we have monoids and modules, we also have extension and restriction of scalars. It is useful to unwind these definitions as well. For the tensor product monoidal structure, this is simple. Suppose $\alpha\colon p \to p'$ is a map of monoid homomorphisms, where $p\colon R_0 \to R_1$ and $p'\colon R_0' \to R_1'$. If $f\colon M_0 \to M_1$ is a p-module, then the extension of scalars functor sends to f to

$$f \otimes_p p'\colon M_0 \otimes_{R_0} R_0' \to M_1 \otimes_{R_1} R_1'.$$

The restriction of scalars functor sends f to itself, as usual.

For the pushout product monoidal structure, life is a bit more complicated.

Proposition 1.9. *Suppose $\alpha\colon j \to j'$ is a map of Smith ideals in a closed symmetric monoidal category $\mathcal{C}$, where $j\colon I \to R$ and $j'\colon I' \to R'$. Let U denote the restriction of scalars functor from j'-modules to j-modules. If N is a j'-module, then $(UN)_0 = N_0$, $(UN)_1 = N_1$, and $f_{UN} = f_N$, where N_0 and N_1 are R-modules via restriction of scalars, and ϕ_{UN} is the composite*

$$N_1 \otimes_R I \xrightarrow{1\otimes\alpha} N_1 \otimes_{R'} I' \xrightarrow{\phi_N} N_0.$$

If M is a j-module, then the extension of scalars $M \,\Box_j\, j'$ has $(M \,\Box_j\, j')_1 = M_1 \otimes_R R'$, and $(M \,\Box_j\, j')_0$ is the pushout in the diagram below.

$$\begin{tikzcd}
(M_0 \otimes_R I') \amalg (M_1 \otimes_R I \otimes_R R') \arrow[r] \arrow[d] & M_0 \otimes_R R' \arrow[d] \\
M_1 \otimes_R I' \arrow[r, "\phi_{M \Box_j j'}"'] & (M \,\Box_j\, j')_0
\end{tikzcd}$$

The map $f_{M\Box_j j'}$ is the evident one, induced by $f_M \otimes 1$ and $1 \otimes j'$.

The corestriction $\operatorname{Hom}_{\Box, j}(j', M)$ has a dual description, using a pullback instead of a pushout, but we leave the details to the reader.

Proof. The only thing we need to prove is the description of $M \,\Box_j\, j'$, which by definition is the coequalizer of the two maps

$$M \,\Box\, j \,\Box\, j' \rightrightarrows M \,\Box\, j'.$$

Both Ev_0 and Ev_1 preserve coequalizers, and it follows easily that $\mathrm{Ev}_1(M \mathbin{\Box}_j j') = M_1 \otimes_R R'$. We have

$$\mathrm{Ev}_0(M \mathbin{\Box} j') = (M_0 \otimes R') \amalg_{M_0 \otimes I'} (M_1 \otimes I'),$$

whereas $\mathrm{Ev}_0(M \mathbin{\Box} j \mathbin{\Box} j')$ is a pushout of the three objects

$$M_0 \otimes R \otimes R', M_1 \otimes I \otimes R', \text{ and } M_1 \otimes R \otimes I'.$$

Upon taking the coequalizer, the first of these objects converts $M_0 \otimes R'$ to $M_0 \otimes_R R'$, the second appears in our description (suitably tensored over R), and the third converts $M_1 \otimes I'$ to $M_1 \otimes_R I'$. □

We now return to the cokernel functor, which we recall from Theorem 1.4 is a symmetric monoidal functor from the pushout product monoidal structure to the tensor product monoidal structure. It therefore preserves monoids and modules. More precisely, we have the following theorem.

Theorem 1.10. *Suppose $\mathcal{C}$ is a pointed closed symmetric monoidal category. The cokernel induces a functor from Smith ideals in $\mathcal{C}$ to monoid homomorphisms in $\mathcal{C}$ whose right adjoint is the kernel. That is, if $j\colon I \to R$ is a Smith ideal, then the cokernel $R \to R/I$ is canonically a monoid homomorphism. Furthermore, the cokernel also induces a functor from modules over the Smith ideal j to modules over the monoid homomorphism* $\operatorname{coker} j$ *whose right adjoint is the kernel.*

Proof. This is an immediate corollary of Theorem 1.4. Since the cokernel is (strongly) symmetric monoidal, its right adjoint, the kernel, is lax symmetric monoidal. That is, there is a natural map

$$\ker f \mathbin{\Box} \ker g \to \ker(f \otimes g)$$

adjoint to the map

$$\operatorname{coker}(\ker f \mathbin{\Box} \ker g) \cong \operatorname{coker} \ker f \otimes \operatorname{coker} \ker g \to f \otimes g,$$

making all the usual coherence diagrams commute. This makes the kernel functor pass to a functor of monoids, where it is right adjoint to the cokernel as a functor of monoids. It also means the kernel defines a functor from $\operatorname{coker} j$-modules to j-modules that is right adjoint to the cokernel as well. This too is standard, but we remind the reader that if f is a $\operatorname{coker} j$-module, then $\ker f$ is a j-module via the multiplication

$$j \mathbin{\Box} \ker f \to \ker \operatorname{coker} j \mathbin{\Box} \ker f \to \ker(\operatorname{coker} j \otimes f) \to \ker f.$$

This completes the proof. □

2. The injective model structure on the arrow category

In this section, we suppose that $\mathcal{C}$ is a closed symmetric monoidal model category in the sense of [Hov99, Definition 4.2.6]. In this case, we would like $\operatorname{Arr} \mathcal{C}$ to be a closed symmetric monoidal model category as well, but we need two different model structures. The model structure compatible with the pushout product is the **projective model structure**, where a map in $\operatorname{Arr} \mathcal{C}$ is a weak equivalence or fibration if and only if its components are so in $\mathcal{C}$. The model structure compatible with the tensor product monoidal structure is the **injective model structure**, where a map in $\operatorname{Arr} \mathcal{C}$ is a weak equivalence or cofibration if and only if its components are so in $\operatorname{Arr} \mathcal{C}$. In this section, we construct the injective model structure and establish the basic properties of it that we need. The main goal is to prove there is a good theory of monoids and modules over them.

Theorem 2.1. *Suppose $\mathcal{C}$ is a model category. Then there is a model structure on* $\operatorname{Arr}\mathcal{C}$*, called the* ***injective model structure****, with the following properties*:

1. *A map α in $\operatorname{Arr}\mathcal{C}$ is a weak equivalence (resp. cofibration) if and only if $\mathrm{Ev}_0\alpha$ and $\mathrm{Ev}_1\alpha$ are weak equivalences (resp. cofibrations) in $\mathcal{C}$;*
2. *A map $\alpha\colon f \to g$ is a (trivial) fibration if and only if the maps $\mathrm{Ev}_1\alpha$ and*
$$\mathrm{Ev}_0 f \xrightarrow{(f,\mathrm{Ev}_0\alpha)} \mathrm{Ev}_1 f \times_{\mathrm{Ev}_1 g} \mathrm{Ev}_0 g$$
are (trivial) fibrations in $\mathcal{C}$. In particular, if α is a fibration in $\operatorname{Arr}\mathcal{C}$, then both $\mathrm{Ev}_0\alpha$ and $\mathrm{Ev}_1\alpha$ are fibrations in $\mathcal{C}$.
3. *The functors $L_0, L_1\colon \mathcal{C} \to \operatorname{Arr}\mathcal{C}$ are left Quillen functors, as are Ev_0 and Ev_1.*
4. *If $\mathcal{C}$ is a symmetric monoidal model category, the tensor product monoidal structure and the injective model structure on $\operatorname{Arr}\mathcal{C}$ make $\operatorname{Arr}\mathcal{C}$ into a symmetric monoidal model category.*

The injective model structure on $\operatorname{Arr}\mathcal{C}$ can now be used as input into Theorem 2.1 to obtain a doubly injective model structure on $\operatorname{Arr}\operatorname{Arr}\mathcal{C}$, if desired.

Proof. We think of the category $\mathcal{J}$ with two objects and one non-identity map as an inverse category in the sense of [Hov99, Definition 5.1.1]. Then [Hov99, Theorem 5.1.3] implies parts (1) and (2), since $\operatorname{Arr}\mathcal{C}$ is the category of $\mathcal{J}$-diagrams in $\mathcal{C}$. We also need the dual of [Hov99, Remark 5.1.7] to see that if α is a fibration, then $\mathrm{Ev}_0\alpha$ is a fibration in $\mathcal{C}$. Part (3) follows, since the functors Ev_i preserve weak equivalences, fibrations, and cofibrations.

For part (4), suppose that $\alpha\colon f \to g$ and $\beta\colon f' \to g'$ are cofibrations in the injective model structure. We must show that the map
$$(f \otimes g') \amalg_{f\otimes f'} (f' \otimes g) \to g \otimes g'$$
is a cofibration, which is trivial if either α or β is. But the components of this map are precisely
$$\mathrm{Ev}_0\alpha \,\square\, \mathrm{Ev}_0\beta \text{ and } \mathrm{Ev}_1\alpha \,\square\, \mathrm{Ev}_1\beta$$
and these are guaranteed to be cofibrations, trivial if either α or β is so, by the fact that $\mathcal{C}$ is a monoidal model category. We recall that there is also a unit condition; we need to know that the map $Q1_S \otimes \alpha \to \alpha$ is a weak equivalence for all cofibrant α, where $Q1_S$ is a cofibrant replacement of the unit 1_S of the tensor product monoidal structure. In fact, if QS is a cofibrant replacement of S in $\mathcal{C}$, then 1_{QS} is a cofibrant replacement of 1_S in the injective model structure. Hence the unit axiom for $\operatorname{Arr}\mathcal{C}$ follows from the unit axiom for $\mathcal{C}$. □

To have a really good theory of monoids and modules over them in a symmetric monoidal model category, though, we need to know a bit more about the model structure. The monoid axiom [SS00, Definition 2.2] guarantees that there is an induced model structure on monoids and on modules over them.

Proposition 2.2. *Let $\mathcal{C}$ be a model category, and give $\operatorname{Arr}\mathcal{C}$ the injective model structure.*

1. *If $\mathcal{C}$ is cofibrantly generated, so is $\operatorname{Arr}\mathcal{C}$.*

(2) *If $\mathcal{C}$ is a symmetric monoidal model category and satisfies the monoid axiom of* [SS00, Definition 2.2]*, so does* $\operatorname{Arr}\mathcal{C}$*.*

Proof. Suppose $\mathcal{C}$ is cofibrantly generated, with generating cofibrations I and generating trivial cofibrations J. Let $I' \subseteq \operatorname{mor}\operatorname{Arr}\mathcal{C}$ consist of the maps $L_1 i$ for $i \in I$ and

$$\alpha_i \colon i \to U_1 \mathrm{Ev}_1 i$$

for $i \in I$. If $i \colon A \to B$, then α_i is the map

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle i}\big\downarrow & & \big\| \\ B & = & B. \end{array}$$

in $\operatorname{Arr}\mathcal{C}$. Define J' similarly using J. Then one can check that β has the right lifting property with respect to $L_1 i$ if and only if $\mathrm{Ev}_1\beta$ has the right lifting property with respect to i, and $\beta \colon f \to g$ has the right lifting property with respect to α_i if and only if

$$\mathrm{Ev}_0 f \to \mathrm{Ev}_1 f \times_{\mathrm{Ev}_1 g} \mathrm{Ev}_0 g$$

has the right lifting property with respect to i. Therefore the maps I' and J' will serve as generating cofibrations and generating trivial cofibrations for the injective model structure. One must also check that the domains and codomains of I' and J' are small in $\operatorname{Arr}\mathcal{C}$, but this follows from smallness in $\mathcal{C}$.

The monoid axiom says that transfinite compositions of pushouts of maps of the form $f \otimes A$, where f is a trivial cofibration in $\mathcal{C}$ and A is an object of $\mathcal{C}$, are weak equivalences. So in $\operatorname{Arr}\mathcal{C}$ we must check that transfinite compositions β of pushouts of maps of the form $\alpha \otimes g$, where α is a trivial cofibration and g is an object of $\operatorname{Arr}\mathcal{C}$, are weak equivalences. Since Ev_i commutes with transfinite compositions and pushouts for $i = 0, 1$, $\mathrm{Ev}_i\beta$ is a transfinite composition of pushouts of maps of the form

$$\mathrm{Ev}_i(\alpha \otimes g) = \mathrm{Ev}_i\alpha \otimes \mathrm{Ev}_i g.$$

Since $\mathrm{Ev}_i\alpha$ is a trivial cofibration in $\mathcal{C}$, $\mathrm{Ev}_i\beta$ is a weak equivalence by the monoid axiom in $\mathcal{C}$. Therefore, β is a weak equivalence as required. $\square$

Corollary 2.3. *Suppoce $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom.*

(1) *There is a model structure on the category of monoid homomorphisms in $\mathcal{C}$, in which a map $\alpha \colon f \to f'$ is a weak equivalence or fibration if and only if it is so in the injective model structure on* $\operatorname{Arr}\mathcal{C}$*. In particular, α is a weak equivalence if and only if* $\mathrm{Ev}_0\alpha$ *and* $\mathrm{Ev}_1\alpha$ *are weak equivalences in $\mathcal{C}$.*
(2) *Given a monoid homomorphism $f \colon R_0 \to R_1$, there is a model structure on the category of f-modules in which α is a weak equivalence or fibration if and only if it is so in the injective model structure on* $\operatorname{Arr}\mathcal{C}$*. In particular, α is a weak equivalence if and only if* $\mathrm{Ev}_0\alpha$ *and* $\mathrm{Ev}_1\alpha$ *are weak equivalences in $\mathcal{C}$.*

This corollary follows immediately from [SS00, Theorem 3.1]. Note that a monoid homomorphism f is fibrant if and only if it is a fibration of fibrant objects in $\mathcal{C}$.

But we would also like a weak equivalence of monoids to induce a corresponding Quillen equivalence of the categories of modules, as in [SS00, Theorem 3.3], so that the module categories are homotopy invariant. This requires a bit more.

Definition 2.4. Suppose $\mathcal{C}$ is a symmetric monoidal model category. If R is a monoid in $\mathcal{C}$ and M is a right R-module, we say that M is **flat** over R if the functor $M \otimes_R (-)$ takes weak equivalences of left R-modules to weak equivalences in $\mathcal{C}$. Here a weak equivalence of left R-modules is an R-module map that is a weak equivalence in $\mathcal{C}$. We say that **cofibrant modules are flat** in $\mathcal{C}$ if, for every monoid R, every cofibrant right R-module M is flat over R.

Then Theorem 3.3 of [SS00] says that if cofibrant modules are flat in $\mathcal{C}$, and $\mathcal{C}$ satisfies the monoid axiom, then module categories are homotopy invariant.

Proposition 2.5. *Suppose $\mathcal{C}$ is cofibrantly generated and satisfies the monoid axiom, and cofibrant modules are flat in $\mathcal{C}$. Then cofibrant modules are flat in the injective model structure on* $\operatorname{Arr}\mathcal{C}$*. In particular, in this case a weak equivalence $\alpha \colon f \to g$ of monoid homomorphisms induces a Quillen equivalence of the corresponding model categories of f-modules and g-modules.*

Before proving this proposition, we need a lemma.

Lemma 2.6. *Suppose $\mathcal{C}$ is cofibrantly generated and satisfies the monoid axiom, and $f \colon R_0 \to R_1$ is a monoid in the tensor product monoidal structure. Then* $\mathrm{Ev}_0 \colon$ *f-mod $\to$ R_0-mod and* $\mathrm{Ev}_1 \colon$ *f-mod $\to$ R_1-mod are left and right Quillen functors.*

Proof. Since Ev_0 and Ev_1 are strict monoidal as functors from the tensor product monoidal structure to $\mathcal{C}$, they induce functors from the module categories as desired, as do their right adjoints U_0 and U_1. The left adjoint L_1 is also strict monoidal and so induces a functor $L_1 \colon R_1$-mod $\to f$-mod left adjoint to Ev_1. However, L_0 is not monoidal, and the left adjoint of $\mathrm{Ev}_0 \colon f$-mod $\to R_0$-mod is instead L_0', where $L_0'(A)$ is the map $A \to R_1 \otimes_{R_0} A$.

Certainly Ev_0 and Ev_1 preserve weak equivalences. A fibration of f-modules in particular has $\mathrm{Ev}_0 f$ and $\mathrm{Ev}_1 f$ fibrations in $\mathcal{C}$, and so also in R_0-mod and R_1-mod, respectively. Thus Ev_0 and Ev_1 are right Quillen functors. To see they are also left Quillen functors, take a (trivial) fibration h of R-modules. Since $U_0 A$ is just the map $A \to 0$, one can check easily that $U_0 h$ is a (trivial) fibration of f-modules. Similarly, if h is a (trivial) fibration of R'-modules, use the fact that $U_1 A = 1_A$ to see that $U_1 h$ is a (trivial) fibration of f-modules. □

Proof of Proposition 2.5. Suppose f is a monoid in the tensor product monoidal structure, so that $f \colon R_0 \to R_1$ is a homomorphism of monoids in $\mathcal{C}$. Let g be a cofibrant f-module. Then $\mathrm{Ev}_0 g$ is a cofibrant R_0-module and $\mathrm{Ev}_1 g$ is a cofibrant R_1-module by Lemma 2.6. Now suppose α is a weak equivalence of left f-modules, so that $\mathrm{Ev}_0 \alpha$ is a weak equivalence of R_0-modules and $\mathrm{Ev}_1 \alpha$ is a weak equivalence of R_1-modules. Then

$$\mathrm{Ev}_0(g \otimes_f \alpha) = \mathrm{Ev}_0 g \otimes_{R_0} \mathrm{Ev}_0 \alpha \text{ and } \mathrm{Ev}_1(g \otimes_f \alpha) = \mathrm{Ev}_1 g \otimes_{R_1} \mathrm{Ev}_1 \alpha$$

so the result follows from the fact that cofibrant modules are flat in $\mathcal{C}$. □

3. The projective model structure on the arrow category

In this section, we establish the projective model structure on $\operatorname{Arr}\mathcal{C}$, which is compatible with the pushout product monoidal structure. Just as in the previous section, we show that there is a good theory of monoids (Smith ideals) and modules over them, although stronger assumptions on $\mathcal{C}$ are needed to get a really good theory. In fact, to check that cofibrant modules are flat in the projective model structure is sufficiently complicated that we discuss it in an appendix.

Theorem 3.1. *Suppose $\mathcal{C}$ is a model category. Then there is a model structure on $\operatorname{Arr}\mathcal{C}$, called the* ***projective model structure****, with the following properties*:

1. *A map α in $\operatorname{Arr}\mathcal{C}$ is a weak equivalence (resp. fibration) if and only if $\mathrm{Ev}_0\alpha$ and $\mathrm{Ev}_1\alpha$ are weak equivalences (resp. fibrations) in $\mathcal{C}$;*
2. *A map $\alpha\colon f \to g$ is a (trivial) cofibration if and only if the maps $\mathrm{Ev}_0\alpha$ and*
$$\mathrm{Ev}_1 f \amalg_{\mathrm{Ev}_0 f} \mathrm{Ev}_0 g \xrightarrow{(\mathrm{Ev}_1\alpha, g)} \mathrm{Ev}_1 g$$
are (trivial) cofibrations in $\mathcal{C}$. In particular, if α is a cofibration in $\operatorname{Arr}\mathcal{C}$, then both $\mathrm{Ev}_0\alpha$ and $\mathrm{Ev}_1\alpha$ are cofibrations in $\mathcal{C}$.
3. *The functors $L_0, L_1 \to \mathcal{C} \to \operatorname{Arr}\mathcal{C}$ are left Quillen functors, as are Ev_0 and Ev_1.*
4. *If $\mathcal{C}$ is cofibrantly generated, so is the projective model structure on $\operatorname{Arr}\mathcal{C}$.*
5. *If $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category, the pushout product monoidal structure and the projective model structure on $\operatorname{Arr}\mathcal{C}$ make $\operatorname{Arr}\mathcal{C}$ into a symmetric monoidal model category.*

The reader may well object that part (5) of the theorem above should not require the cofibrantly generated hypothesis. This is surely correct, but the proof involves so many pushout diagrams of pushout diagrams that the author got too confused to finish the proof.

The reader should note that the identity functor is a Quillen equivalence from the projective model structure to the injective model structure on $\operatorname{Arr}\mathcal{C}$.

At least if $\mathcal{C}$ is cofibrantly generated, the projective model structure can also be iterated, and we could have the doubly projective model structure on $\operatorname{Arr}\operatorname{Arr}\mathcal{C}$, in addition to the projective injective model structure and the injective projective model structure. We don't know if these are useful.

Proof. We think of the category $\mathcal{J}$ with two objects and one non-identity map as a direct category in the sense of [Hov99, Definition 5.1.1]. Then [Hov99, Theorem 5.1.3] implies parts (1) and (2), since $\operatorname{Arr}\mathcal{C}$ is the category of $\mathcal{J}$-diagrams in $\mathcal{C}$. Then [Hov99, Remark 5.1.7] tells us that if α is a cofibration in the projective model structure, $\mathrm{Ev}_1\alpha$ (and also $\mathrm{Ev}_1\alpha$, of course) is a cofibration in $\mathcal{C}$. Part (3) now follows easily, since the functors Ev_i preserve weak equivalences, fibrations, and cofibrations.

For part (4), suppose $\mathcal{C}$ is cofibrantly generated, with generating cofibrations I and generating trivial cofibrations J. Then $\operatorname{Arr}\mathcal{C}$ is cofibrantly generated with generating cofibrations $L_0 I \cup L_1 I$ and generating trivial cofibrations $L_0 J \cup L_1 J$, by [Hov99, Remark 5.1.8].

We must now verify that $\operatorname{Arr}\mathcal{C}$ is a symmetric monoidal model category when $\mathcal{C}$ is so (and is cofibrantly generated). We will check the unit condition below. For the remaining condition, because $\operatorname{Arr}\mathcal{C}$ is cofibrantly generated, we just have to check that if α is a generating cofibration and β is a generating (trivial) cofibration of $\operatorname{Arr}\mathcal{C}$, then the pushout product $\alpha \,\square_2\, \beta$ (not to be confused with the monoidal structure $\square$ in $\operatorname{Arr}\mathcal{C}$) is a (trivial) cofibration in $\operatorname{Arr}\mathcal{C}$. Here, if $\alpha\colon f \to g$ and $\beta\colon f' \to g'$,
$$\alpha \,\square_2\, \beta\colon (f \,\square\, g') \amalg_{f \square f'} (g \,\square\, f') \to g \,\square\, g'.$$

Now, if $\alpha = L_0 i\colon L_0 A \to L_0 B$, it follows from Lemma 1.3 and some computation that

$$L_0 i \,\square_2\, \beta = L_0(i \,\square\, \mathrm{Ev}_1 \beta).$$

In particular, if i is a cofibration in $\mathcal{C}$ and β is a (trivial) cofibration in $\operatorname{Arr}\mathcal{C}$, then $\mathrm{Ev}_1\beta$ is a (trivial) cofibration in $\mathcal{C}$, so $i \,\square\, \mathrm{Ev}_1\beta$ is a (trivial) cofibration in $\mathcal{C}$, and thus $L_0 i \,\square_2\, \beta$ is a (trivial) cofibration in $\operatorname{Arr}\mathcal{C}$.

This takes care of all of the $\alpha \,\square_2\, \beta$ we need to consider except $\alpha = L_1 i$ and $\beta = L_1 j$. Since L_1 is symmetric monoidal, we have

$$L_1 i \,\square_2\, L_1 j = L_1(i \,\square\, j).$$

Thus, if i is a cofibration in $\mathcal{C}$ and j is a (trivial) cofibration in $\mathcal{C}$, then $i \,\square\, j$ is a (trivial) cofibration in $\mathcal{C}$, so $L_1 i \square_2 L_1 j$ is a (trivial) cofibration in $\operatorname{Arr}\mathcal{C}$, as required.

We are now left with checking the unit condition. Recall that the unit condition in $\mathcal{C}$ says that, if $QS \to S$ is a cofibrant replacement for the unit S, then $QS \otimes A \to A$ is a weak equivalence for all cofibrant A in $\mathcal{C}$. In $\operatorname{Arr}\mathcal{C}$, f is a cofibrant object if and only if f is a cofibration of cofibrant objects of $\mathcal{C}$. It follows that $L_1(QS)\colon\ 0 \to QS$ is a cofibrant replacement of the unit $L_1 S$. It follows from Lemma 1.3 that

$$L_1(QS) \,\square\, f \to f = QS \otimes f \to f.$$

In particular, if f is cofibrant, the domain and codomain of f are cofibrant in $\mathcal{C}$, so the unit axiom in $\mathcal{C}$ implies this map is a weak equivalence in $\operatorname{Arr}\mathcal{C}$, as required. $\square$

Just as with the injective model structure, we would like to ensure that good properties of the symmetric monoidal model category $\mathcal{C}$ are inherited by the projective model structure on $\operatorname{Arr}\mathcal{C}$. There is no difficulty with the monoid axiom.

Proposition 3.2. *If $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category that satisfies the monoid axiom, then the projective model structure on* $\operatorname{Arr}\mathcal{C}$ *also satisfies the monoid axiom.*

Proof. According to [SS00, Lemma 2.3], it suffices to check that transfinite compositions of pushouts of maps in $\operatorname{Arr}\mathcal{C}$ of the form $L_0 j \,\square\, f$ and $L_1 j \,\square\, f$ are weak equivalences, where j is a trivial cofibration in $\mathcal{C}$. By Lemma 1.1, we have $L_0 j \,\square\, f = L_0(j \otimes \mathrm{Ev}_1 f)$. Thus

$$\mathrm{Ev}_0(L_0 j \,\square\, f) = \mathrm{Ev}_1(L_0 j \,\square\, f) = j \otimes \mathrm{Ev}_1 f,$$

and this is a trivial cofibration in $\mathcal{C}$ tensored with an object of $\mathcal{C}$. Similarly, Lemma 1.1 tells us that

$$\mathrm{Ev}_0(L_1 j \,\square\, f) = j \otimes \mathrm{Ev}_0 f \text{ and } \mathrm{Ev}_1(L_1 j \,\square\, f) = j \otimes \mathrm{Ev}_1 f.$$

Therefore, these maps are also trivial cofibrations in $\mathcal{C}$ tensored with objects of $\mathcal{C}$. Since Ev_0 and Ev_1 are left adjoints, they commute with transfinite compositions and pushouts. Thus, if we apply Ev_0 or Ev_1 to a transfinite composition of pushouts of maps of the form $L_0 j \,\square\, f$ and $L_1 j \,\square\, f$, we will get a transfinite composition of pushouts of maps of the form $j \otimes X$, which are weak equivalences in $\mathcal{C}$ as required, because $\mathcal{C}$ satisfies the monoid axiom. $\square$

Corollary 3.3. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom.*

(1) *There is a model structure on the category of Smith ideals in $\mathcal{C}$, in which a map $\alpha\colon j \to j'$ is a weak equivalence or fibration if and only if* $\mathrm{Ev}_0\alpha$ *and* $\mathrm{Ev}_1\alpha$ *are weak equivalences or fibrations in $\mathcal{C}$.*
(2) *Given a Smith ideal j, there is a model structure on the category of j-modules in which α is a weak equivalence or fibration if and only if* $\mathrm{Ev}_0\alpha$ *and* $\mathrm{Ev}_1\alpha$ *are weak equivalences or fibrations in $\mathcal{C}$.*

This corollary follows from [SS00, Theorem 3.1].
It is also useful to point out the following fact.

Corollary 3.4. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom.*

(1) *The functor* Ev_1 *is a left Quillen functor from Smith ideals in $\mathcal{C}$ to monoids in $\mathcal{C}$. In particular, the codomain of a cofibrant Smith ideal is a cofibrant monoid.*
(2) *If the unit S is cofibrant in $\mathcal{C}$, then a cofibrant Smith ideal is, in particular, a cofibration of cofibrant objects in $\mathcal{C}$.*

Proof. The functor Ev_1 is strict monoidal with respect to the pushout product monoidal structure, so does induce a functor from Smith ideals to monoids. Its right adjoint U_1 takes R to 1_R, and this obviously perserves weak equivalences and fibrations. Thus Ev_1 is a left Quillen functor.

If the unit S is cofibrant in $\mathcal{C}$, then the unit $0 \to S$ of the pushout product monoidal structure is cofibrant in $\mathrm{Arr}\,\mathcal{C}$, and so a cofibrant Smith ideal is in particular cofibrant in the projective model structure on $\mathrm{Arr}\,\mathcal{C}$ by [SS00, Theorem 3.1]. □

We would also like to know that if cofibrant modules are flat in $\mathcal{C}$ (see Definition 2.4), then cofibrant modules are flat in the projective model structure on $\mathrm{Arr}\,\mathcal{C}$. This seems to be more complicated than the corresponding question for the injective module structure, so we postpone this discussion to the appendix, where we will prove the following theorem. The definition of a pure class of morphisms is Definition A.1.

Theorem 3.5. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom, and $\mathcal{P}$ is a pure class of morphisms of $\mathcal{C}$ containing the maps $i \otimes X$ for all generating cofibrations i of $\mathcal{C}$ and all $X \in \mathcal{C}$. Assume that the domains and codomains of the generating cofibrations of $\mathcal{C}$ are flat. Then cofibrant modules are flat in the projective model structure on* $\mathrm{Arr}\,\mathcal{C}$*. In this case, a weak equivalence of Smith ideals induces a Quillen equivalence of the corresponding module categories.*

4. The cokernel functor in the stable case

In this section, we prove that the cokernel functor is a left Quillen functor from the projective model structure on $\mathrm{Arr}\,\mathcal{C}$ to the injective model structure. Even better, when $\mathcal{C}$ is stable, in the sense that $\mathrm{ho}\,\mathcal{C}$ is triangulated, the cokernel functor is a Quillen equivalence. Thus a Smith ideal is the same thing as a monoid homomorphism, up to homotopy.

Proposition 4.1. *Suppose $\mathcal{C}$ is a pointed model category. The cokernel is a left Quillen functor from the projective model structure on* $\mathrm{Arr}\,\mathcal{C}$ *to the injective model structure.*

Proof. Consider the category $\mathcal{J}$ with three objects $-1, 0, 1$ and two non-identity morphisms $0 \to 1$ and $0 \to -1$. We consider the first map as raising degree, and the second as lowering degree. This makes our category a Reedy category [Hov99, Definition 5.2.1]. There is therefore a model structure on $\mathcal{J}$-diagrams on $\mathcal{C}$ [Hov99, Theorem 5.2.5]. This is the model structure used in the proof of the cube lemma [Hov99, Lemma 5.2.6], where it is proved that the colimit, which is just the pushout, is a left Quillen functor from $\mathcal{C}^{\mathcal{J}}$ to $\mathcal{C}$.

Now suppose $\alpha \colon f \to g$ is a (trivial) cofibration in the projective model structure. Write $f \colon X_0 \to X_1$ and $g \colon Y_0 \to Y_1$. We then have the associated objects of $\mathcal{C}^{\mathcal{J}}$, namely

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ \downarrow & & \\ 0 & & \end{array}$$

and

$$\begin{array}{ccc} Y_0 & \xrightarrow{g} & Y_1 \\ \downarrow & & \\ 0. & & \end{array}$$

The map α induces a map between these diagrams, and this map is a (trivial) cofibration in $\mathcal{C}^{\mathcal{J}}$. Indeed, because the map $0 \to -1$ in $\mathcal{J}$ lowers degree, we just need

$$1_0 \colon 0 \to 0, \alpha_0 \colon X_0 \to Y_0 \text{ and } (\alpha_1, g) \colon X_1 \amalg_{X_0} Y_0 \to Y_1$$

to be (trivial) cofibrations, as of course they are. We conclude that the induced map

$$\operatorname{coker} f \to \operatorname{coker} g$$

of the colimits is a (trivial) cofibration, which is just what we need to make coker a left Quillen functor from the projective to the injective model structure. □

Corollary 4.2. *Suppose $\mathcal{C}$ is a cofibrantly generated pointed symmetric monoidal model category.*

(1) *The cokernel is a left Quillen functor from the model category of Smith ideals in $\mathcal{C}$ to the model cateory of monoid homomorphisms in $\mathcal{C}$.*
(2) *If j is a Smith ideal in $\mathcal{C}$, the cokernel induces a left Quillen functor from the model category of j-modules to the model category of* $\operatorname{coker} j$*-modules.*

Proof. This follows from the general theory. The right adjoint ker preserves (trivial) fibrations in $\operatorname{Arr} \mathcal{C}$, and since (trivial) fibrations of monoids or modules are just maps of monoids or modules that are (trivial) fibrations in $\operatorname{Arr} \mathcal{C}$, ker will preserve (trivial) fibrations of monoids and modules. □

We now suppose that $\mathcal{C}$ is **stable**, so that $\operatorname{ho} \mathcal{C}$ is a triangulated category [Hov99, Chapter 7]. Since a map of exact triangles in a triangulated category that is an isomorphism on two of the three spots is an isomorphism on the third spot as well, we should expect the cokernel to be a Quillen equivalence in this case. This is in fact true.

Theorem 4.3. *Suppose $\mathcal{C}$ is a stable model category. Then the cokernel is a Quillen equivalence from the projective model structure on* $\operatorname{Arr} \mathcal{C}$ *to the injective model structure.*

Proof. Suppose f is cofibrant in the projective model structure on $\operatorname{Arr}\mathcal{C}$ and p is fibrant in the injective model structure. This means that $f\colon A \to B$ is a cofibration of cofibrant objects and $p\colon X \to Y$ is a fibration of fibrant objects. Let $\alpha\colon \operatorname{coker} f \to p$ be a map, which we write

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\alpha_0"'] & \operatorname{coker} f \arrow[d, "\alpha_1"] \\ X \arrow[r, "p"'] & Y, \end{tikzcd}$$

with corresponding map $\beta\colon f \to \ker p$, written

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\beta_0"'] & B \arrow[d, "\alpha_0"] \\ \ker p \arrow[r, "q"'] & X. \end{tikzcd}$$

We must show that a map α is a weak equivalence if and only if β is. In the homotopy category $\operatorname{ho}\mathcal{C}$, the map f gives rise to a cofiber sequence

$$A \xrightarrow{f} B \to \operatorname{coker} f \to \Sigma A$$

and the map g gives rise to a fiber sequence

$$\Omega Y \to \ker p \xrightarrow{q} X \xrightarrow{p} Y.$$

Since $\mathcal{C}$ is stable, every fiber sequence is (canonically isomorphic to) a cofiber sequence, and so

$$\ker p \xrightarrow{q} X \xrightarrow{p} Y \to \Sigma(\ker p)$$

is a cofiber sequence, Then α and β together define a map between two exact triangles in a triangulated category; if α or β is a weak equivalence, then this map is an isomorphism at two out of the three spots, so also an isomorphism at the third spot. Thus α is a weak equivalence if and only if β is so. □

Corollary 4.4. *Suppose $\mathcal{C}$ is a cofibrantly generated stable symmetric monoidal model category in which the unit S is cofibrant.*

(1) *The cokernel is a Quillen equivalence from the model category of Smith ideals in $\mathcal{C}$ to the model category of monoid homomorphisms in $\mathcal{C}$.*
(2) *If j is a cofibrant Smith ideal in $\mathcal{C}$, the cokernel induces a Quillen equivalence from the model category of j-modules to the model category of* $\operatorname{coker} j$*-modules.*

Proof. The kernel, as a functor from monoid homomorphisms to Smith ideals, reflects weak equivalences between fibrant objects. Indeed, since fibrations and weak equivalences are created in the underlying model structures on $\operatorname{Arr}\mathcal{C}$, this follows from fact that the kernel, as the right half of a Quillen equivalence on $\operatorname{Arr}\mathcal{C}$, reflects weak equivalences between fibrant objects [Hov99, Corollary 1.3.16].

Now suppose j is a cofibrant Smith ideal. Since S is cofibrant, the unit $L_1 S$ of the pushout product monoidal structure is cofibrant in the projective model structure. Hence a cofibrant monoid is in particular cofibrant in $\operatorname{Arr}\mathcal{C}$ by [SS00, Theorem 3.1]. Thus j is cofibrant in $\operatorname{Arr}\mathcal{C}$. This means that $j \to \ker T(\operatorname{coker} j)$ is a weak equivalence, where T is a fibrant replacement functor in the injective model structure on $\operatorname{Arr}\mathcal{C}$, again using [Hov99, Corollary 1.3.16]. If T' denotes a fibrant replacement functor in the category of monoid homomorphisms, there is a weak equivalence $T(\operatorname{coker} j) \to T'(\operatorname{coker} j)$ in the injective model structure. These are both fibrant objects, and so the induced map $\ker T(\operatorname{coker} j) \to \ker T'(\operatorname{coker} j)$ is a weak equivalence. Thus the map $j \to \ker T'(\operatorname{coker} j)$ is a weak equivalence, and so the cokernel is a Quillen equivalence from Smith ideals to monoid homomorphisms by [Hov99, Corollary 1.3.16].

Now again assume j is a cofibrant Smith ideal. The kernel again reflects weak equivalences between fibrant $\operatorname{coker} j$-modules. We claim that a cofibrant j-module is also cofibrant in $\mathcal{C}$, so that we can repeat the above argument to complete the proof that the cokernel is a Quillen equivalence from j-modules to Fj-modules. Indeed, a cofibrant object in any cofibrantly generated model category is a transfinite extension of the cokernels of the generating cofibrations. In our case, these cokernels are of the form $j \mathbin{\square} q$, where q is a cokernel of a generating cofibration in $\operatorname{Arr}\mathcal{C}$ and hence cofibrant in $\operatorname{Arr}\mathcal{C}$. Since j is also cofibrant in $\operatorname{Arr}\mathcal{C}$, we conclude that cofibrant j-modules are cofibrant in $\operatorname{Arr}\mathcal{C}$. $\square$

Appendix A. Pure classes of morphisms

In this appendix, we prove Theorem 3.5 about when cofibrant modules are flat in the projective model structure on $\operatorname{Arr}\mathcal{C}$.

Consider the more general question of when cofibrant R-modules are flat, for R a monoid in a symmetric monoidal model category $\mathcal{C}$. The logical way to approach this is to build up from the generating cofibrations in R-mod to all cofibrant R-modules. So we should have some result that asserts that if the domains and codomains of the generating cofibrations in $\mathcal{C}$ are flat, then all cofibrant R-modules are flat, for any R. A cofibrant R-module is a retract of a transfinite composition of pushouts of maps $i \otimes 1\colon A \otimes R \to B \otimes R$, where $i\colon A \to B$ is a generating cofibration of $\mathcal{C}$. In general, pushouts of maps in model categories only behave well when the maps are cofibrations, and we cannot expect $i \otimes 1$ to be a cofibration. But it might be something a little weaker.

Definition A.1. Define a class of morphisms $\mathcal{P}$ in a model category $\mathcal{C}$ to be a **pure class** if the following properties hold.

(1) A pushout of a map in $\mathcal{P}$ is in $\mathcal{P}$.
(2) If we have a map of diagrams

$$\begin{tikzcd} B \arrow[d] & A \arrow[l] \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l] \arrow[r, "f'"'] & C' \end{tikzcd}$$

in which the vertical maps are weak equivalences, and f and f' are in $\mathcal{P}$, then the induced map of pushouts

$$B \amalg_A C \to B' \amalg_{A'} C'$$

is a weak equivalence.
(3) If λ is an ordinal, $X, Y\colon \lambda \to \mathcal{C}$ are colimit-preserving functors such that each map $X_\alpha \to X_{\alpha+1}$ and $Y_\alpha \to Y_{\alpha+1}$ is in $\mathcal{P}$, and $f\colon X \to Y$ is a natural

transformation such that f_α is a weak equivalence for all $\alpha < \lambda$, then the induced map
$$\operatorname{colim}_{\alpha<\lambda} X_\alpha \to \operatorname{colim}_{\alpha<\lambda} Y_\alpha$$
is a weak equivalence.

The basic example of a pure class of morphisms is the class of cofibrations in a left proper model category, by [Hir03, Proposition 13.5.4] and [Hir03, Proposition 17.9.3].

The main advantage of a pure class of morphisms is the following theorem.

Theorem A.2. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom, and $\mathcal{P}$ is a pure class of morphisms of $\mathcal{C}$ containing the maps $i \otimes X$ for all generating cofibrations i of $\mathcal{C}$ and all $X \in \mathcal{C}$. Assume that the domains and codomains of the generating cofibrations of $\mathcal{C}$ are flat. Then cofibrant R-modules are flat, for any monoid R in $\mathcal{C}$.*

Proof. Since retracts of flat R-modules are flat, we can assume our cofibrant R-module $X = \operatorname{colim}_{\alpha<\lambda} X_\alpha$ for some ordinal λ and some colimit-preserving functor, where each map $X_\alpha \to X_{\alpha+1}$ is a pushout of a generating cofibration of R-mod and $X_0 = 0$. We prove that X_α is flat for all $\alpha \leq \lambda$ by transfinite induction, the base case $X_0 = 0$ being obvious. Let $f\colon M \to N$ be a weak equivalence of left R-modules.

For the successor ordinal case, we have a pushout
$$\begin{array}{ccc} A\otimes R & \xrightarrow{i\otimes 1} & B\otimes R \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & X_{\alpha+1} \end{array}$$
where i is a generating cofibration of $\mathcal{C}$. We then get a map of pushout squares from
$$\begin{array}{ccc} A\otimes M & \longrightarrow & B\otimes M \\ \downarrow & & \downarrow \\ X_\alpha \otimes_R M & \longrightarrow & X_{\alpha+1}\otimes_R M \end{array}$$
to
$$\begin{array}{ccc} A\otimes N & \longrightarrow & B\otimes N \\ \downarrow & & \downarrow \\ X_\alpha \otimes_R N & \longrightarrow & X_{\alpha+1}\otimes_R N. \end{array}$$
Since A and B are flat in $\mathcal{C}$, and X_α is a a flat R-module by the induction hypothesis, this map is a weak equivalence at all of the corners except the bottom-right corner. Since the maps $i\otimes M$ and $i\otimes N$ are in the pure class $\mathcal{P}$, we conclude that $X_{\alpha+1}\otimes_R f$ is a weak equivalence as well, concluding the successor ordinal case. However, we also note that
$$X_\alpha \otimes_R M \to X_{\alpha+1}\otimes_R M,$$
and the analogous map for N, is in $\mathcal{P}$.

For the limit ordinal case, suppose X_α is flat for all $\alpha < \beta$. We have a map of diagrams
$$X_\alpha \otimes_R M \to X_\alpha \otimes_R N$$

that is a weak equivalence for all $\alpha < \beta$, and furthermore each map $X_\alpha \otimes_R M \to X_{\alpha+1} \otimes_R M$ is in $\mathcal{P}$, as is the corresponding map for N. Thus

$$X_\beta \otimes_R M \to X_\beta \otimes_R N$$

is a weak equivalence, since $\mathcal{P}$ is a pure class, completing the induction. $\square$

Now we return to Arr. For this we need to transfer our pure class from $\mathcal{C}$ to $\operatorname{Arr}\mathcal{C}$. Fortunately this is straightforward, and we get the following theorem.

Theorem A.3. *Suppose $\mathcal{C}$ is a cofibrantly generated symmetric monoidal model category satisfying the monoid axiom, and $\mathcal{P}$ is a pure class of morphisms of $\mathcal{C}$ containing the maps $i \otimes X$ for all generating cofibrations i of $\mathcal{C}$ and all $X \in \mathcal{C}$. Assume that the domains and codomains of the generating cofibrations of $\mathcal{C}$ are flat. Then cofibrant modules are flat in the projective model structure on* $\operatorname{Arr}\mathcal{C}$*.*

Proof. The plan is to apply Theorem A.2 to $\operatorname{Arr}\mathcal{C}$. The generating cofibrations of $\operatorname{Arr}\mathcal{C}$ consist of the maps $L_0 i$ and $L_1 i$ for generating cofibrations $i\colon A \to B$ of $\mathcal{C}$. So our first job is to check that L_0 and L_1 preserve flat objects. Suppose $\alpha\colon f \to g$ is a weak equivalence in $\operatorname{Arr}\mathcal{C}$. Applying Lemma 1.1, we find that

$$\operatorname{Ev}_i(L_0(X) \mathbin{\square} \alpha) = X \otimes \operatorname{Ev}_1 \alpha$$

and

$$\operatorname{Ev}_i(L_1(X) \mathbin{\square} \alpha) = X \otimes \operatorname{Ev}_i \alpha$$

for $i = 0, 1$. If X is flat, $X \otimes \operatorname{Ev}_i \alpha$ is a weak equivalence, and so both L_0 and L_1 preserve flat objects. We conclude that the domains and codomains of the generating cofibrations of $\operatorname{Arr}\mathcal{C}$ are flat.

Let $\mathcal{Q}$ denote the class of morphisms f in $\operatorname{Arr}\mathcal{C}$ so that $\operatorname{Ev}_0 f$ and $\operatorname{Ev}_1 f$ are in $\mathcal{P}$. Since colimits and weak equivalences in $\operatorname{Arr}\mathcal{C}$ are detected in $\mathcal{C}$, it is easy to see that $\mathcal{Q}$ is a pure class of morphisms in $\operatorname{Arr}\mathcal{C}$. It remains to show that both $L_0 i \mathbin{\square} f$ and $L_1 i \mathbin{\square} f$ are in $\mathcal{Q}$ for i a generating cofibration of $\mathcal{C}$ and f any object of $\operatorname{Arr}\mathcal{C}$. But, again using Lemma 1.1, we find that

$$\operatorname{Ev}_i(L_0 i \mathbin{\square} f) = i \otimes \operatorname{Ev}_1 f$$

and

$$\operatorname{Ev}_i(L_1 i \mathbin{\square} f) = i \otimes \operatorname{Ev}_i f$$

for $i = 0, 1$. Since both of these are in $\mathcal{P}$, we conclude that $L_0 i \mathbin{\square} f$ and $L_1 i \mathbin{\square} f$ are in $\mathcal{Q}$. Hence we can apply Theorem A.2 to $\operatorname{Arr}\mathcal{C}$ to get our theorem. $\square$

References

[AR94] Jiří Adámek and Jiří Rosický, *Locally presentable and accessible categories*, Cambridge University Press, Cambridge, 1994. MR 95j:18001

[BI07] Robert Bruner and Dan Isaksen, *Jeff Smith's theory of ideals*, unpublished, 2007.

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR 2003j:18018

[Hov99] Mark Hovey, *Model categories*, American Mathematical Society, Providence, RI, 1999. MR 99h:55031

[SS00] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511. MR 1734325 (2001c:18006)

Department of Mathematics, Wesleyan University, Middletown, CT 06459
E-mail address: `hovey@member.ams.org`