QUILLEN MODEL CATEGORIES

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Abstract. We provide a brief description of the mathematics that led to Daniel Quillen's introduction of model categories, a summary of his seminal work "Homotopical algebra", and a brief description of some of the developments in the field since.

Introduction

Daniel Quillen's Ph. D. thesis [Qui64], under Raoul Bott at Harvard in 1964, was about partial differential equations. But immediately after that, he moved to MIT and began working on algebraic topology, heavily influenced by Dan Kan. Just three years after his Ph. D., he published a Springer Lecture Notes volume [Qui67] called "Homotopical algebra". This book, in which he introduced model categories, permanently transformed algebraic topology from the study of topological spaces up to homotopy to a general tool useful in many areas of mathematics. Quillen's theory can be applied to almost any situation in which one has a class of maps, called weak equivalences, that one would like to think of as being almost like isomorphisms. This generality has meant that model categories are more important today than they ever been. Without Quillen's work it would be difficult to imagine Voevodsky's Fields Medal winning work on the Milnor conjecture, involving as it does the construction of a convenient stable homotopy category of generalized schemes. The infinity-categories of Joyal, much studied by Lurie and many others, are a direct generalization of model categories.

In this paper we describe some of the influences that led to "Homotopical algebra", some of the structure of the model categories that Quillen introduced, and some of the modern developments in the subject. The reader interested in learning more should definitely turn to [Qui67] first, but could also consult some of the modern introductions to model categories, such as [GJ99], [Hir03], or [Hov99]. Model categories are an active area of research, and we have been able to cite only a few of the many strong results in the field.

The author is honored to write about the work of one of the great masters of his time.

1. Before Quillen

Quillen pulled together several contemporary ideas with his introduction of model categories. The most significant of these were the work of Eilenberg-Zilber,

Date: November 29, 2012.

2010 *Mathematics Subject Classification.* Primary 18G55; Secondary 13D03, 16E35, 55U35.

Key words and phrases. cofibration, derived functor, fibration, homotopical algebra, homotopy limit, model category, Quillen, simplicial set, weak equivalence.

Kan, and Milnor on simplicial sets and the work of Verdier and Grothendieck on derived categories.

Simplicial sets were introduced by Eilenberg and Zilber in [EZ50] as a combinatorial model for topological spaces that would model more general topological spaces than simplicial complexes. As a reminder, a simplicial set X consists of a set of n-simplices K_n for $n \geq 0$, face maps $d_i\colon K_n \to K_{n-1}$ for $0 \leq i \leq n$, and degeneracy maps $s_i\colon K_n \to K_{n+1}$ for $0 \leq i \leq n$. (Simplicial sets without degeneracies were called *semi-simplicial complexes* in [EZ50] and in much of the early literature; what we now call simplicial sets were called *complete semi-simplicial complexes* or c.s.s. complexes). A good example is the unit interval simplicial set, which has two 0-simplices labelled 0 and 1, three 1-simplices 00, 01 (the only non-degenerate 1-simplex), and 11, and, in general, $n+2$ n-simplices. The standard reference for simplicial sets remains [May92].

There are two essential features of simplicial sets that were crucial inputs to Quillen's work. First of all, the evident notion of simplicial homotopy does not behave well for general simplicial sets, but only for simplicial sets that satisfy the extension condition, a property introduced by Kan [Kan56] analogous to injectivity for modules over a ring. These simplicial sets are now called **Kan complexes**, and one must replace a simplicial set by a Kan complex that is in some sense not too different (a **weakly equivalent** Kan complex) before one can calculate homotopy groups and the like.

The other essential feature is the relationship between simplicial sets and topological spaces. A simplicial set K has a **geometric realization** $|K|$ introduced by Milnor [Mil57] and a topological space X has a **singular complex** $\operatorname{Sing} X$. The geometric realization is left adjoint to the singular complex; indeed, Kan invented adjoint functors [Kan58] mainly to explain this example. Kan's extensive work led to Milnor's proof that the map $K \to \operatorname{Sing}|K|$ is a simplicial homotopy equivalence for Kan complexes K and $|\operatorname{Sing} X| \to X$ is a homotopy equivalence for CW complexes X, so that the homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces.

Kan's work also led to Gabriel and Zisman's introduction in [GZ67] of a calculus of fractions for inverting a class of maps in categories, and this was also a major influence on Quillen. In general, if one tries to introduce formal inverses for a collection of maps in a category, the morphisms get out of hand quickly, and it is quite possible to end up with a "category" in which the morphism objects are too big to be sets. The calculus of fractions gives conditions under which it is possible to invert maps and end up with an honest category, and Gabriel and Zisman applied this to simplicial sets and weak equivalences.

The work of Grothendieck and Verdier on the derived category, as written by Hartshorne in [Har66], also influenced Quillen a great deal. Here the idea was to start with chain complexes over an abelian category and invert the homology isomorphisms to obtain the derived category. So again there is a class of weak equivalences one would like to make isomorphisms. This time, however, the focus was much more on derived functors. So, given a functor between abelian categories, Grothendieck and Verdier were interested in how to construct an induced functor between the derived categories. This is not obvious, because only very rare functors, the exact ones, will preserve homology isomorphisms. So Verdier proceeded by first replacing a complex X by a complex with the same homology to which the functor could be applied safely. This is analogous to replacing a module by its projective or injective resolution.

2. Quillen's work

When Quillen wrote "Homotopical Algebra", then, some outlines of what a homotopy theory should be were clear. One should definitely start with a category $\mathcal{C}$ and a collection of morphisms $\mathcal{W}$ called the weak equivalences. The goal should be to invert the weak equivalences to obtain a homotopy category. But there should be a fairly concrete way to do this so as to be able to construct derived functors. For topological spaces X, one has a CW approximation QX equipped with a weak equivalence $QX \to X$, and it was well-understood that one needed to replace X by QX before doing various constructions. This is analogous to replacing a module (or chain complex) by a projective resolution. But for simplicial sets K, one has instead a weak equivalence $K \to RK$, where RK is a Kan complex. Again, one has to replace K by RK for many things—even to define homotopy groups. This is analogous to replacing a chain complex by an injective resolution.

So Quillen was going to need a notion of weak equivalence, a notion of cofibrant object (like QX), and a notion of fibrant object (like RK). Here, however, Quillen had a key insight coming from topology. In non-additive situations, objects don't give enough information, basically because there is no notion of an exact sequence. So Quillen realized he would in fact need two more classes of maps, the **cofibrations** and the **fibrations**. An object is cofibrant if the map from the initial object to it is a cofibration, and dually for fibrant objects.

Naturally these three collections of maps would have to satisfy some axioms. Again, the primary intuition for these axioms came from topology, where a map $p\colon E \to B$ is a **Serre fibration** if and only if for every commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[g]{} & B \end{array}$$

there is a lift $h\colon X \times I \to E$ making both triangles commute. Here X is any finite CW complex, but in fact we could take X to be a disk of arbitrary dimension and we would get the same class of Serre fibrations. We say that p has the **right lifting property with respect to** j and j has the **left lifting property with respect to** p. Quillen generalized this by requiring that every fibration in a model category have the right lifting property with respect to every trivial cofibration (that is, every map that is both a cofibration and a weak equivalence), and every trivial fibration have the right lifting property with respect to every cofibration. This lifting axiom and the factorization axiom (that every map can be factored as a cofibration followed by a trivial fibration and also a trivial cofibration followed by a fibration) are the two essential axioms of a model category.

Given a model category $\mathcal{C}$, Quillen defined a notion of left homotopy and right homotopy for maps, proved these two notions coincide if the source object is cofibrant and the target is fibrant, and showed that the homotopy category is equivalent to the category of cofibrant and fibrant objects and homotopy classes of maps. In particular, the homotopy category does indeed have small Hom-sets. The homotopy category can be obtained from either the full subcategory of cofibrant objects or the full subcategory of fibrant objects by means of the calculus of fractions [GZ67], but not in general from the entire model category. Quillen also developed some extra structure in the homotopy category of a pointed model category, namely the loop and suspension functors, fibration and cofibration sequences, and Toda brackets.

Quillen's main examples were topological spaces, simplicial sets and simplicial algebraic structures, and bounded below chain complexes. We describe these examples below, but the most striking thing about them is that in each one, either every object is cofibrant or every object is fibrant. It was prescient of Quillen to nevertheless carry out his theory in full generality, because that generality has been crucial to the more complicated model categories that have appeared since. For simplicial sets, the cofibrations are monomorphisms, the weak equivalences are the homotopy isomorphisms, and the fibrations are Kan fibrations (and so the fibrant objects are the Kan complexes); thus every object is cofibrant. For simplicial algebraic structures (such as abelian groups, groups, or rings), the fibrations (resp. weak equivalences) are maps of simplicial algebraic structures that are Kan fibrations (resp. weak equivalences) as maps of simplicial sets. The cofibrations are then determined by the lifting axiom; typically a cofibration is obtained by iteratively attaching free algebraic structures (and taking retracts). One might expect then that a simplicial algebraic structure would not in general be fibrant or cofibrant, but in fact Moore [Moo58] had already shown that all simplicial groups are Kan complexes. For topological spaces, the cofibrations are relative cell complexes, the weak equivalences are the homotopy isomorphisms, and the fibrations are the Serre fibrations. All topological spaces are fibrant. For bounded below chain complexes, one can take the cofibrations to be the monomorphisms with degreewise projective cokernel, the weak equivalences to be the homology isomorphisms, and the fibrations to be the surjections, so that all objects are fibrant. Dually, for bounded above chain complexes one can take cofibrations to be monomorphisms, weak equivalences to be homology isomorphisms, and fibrations to be surjections with degreewise injective kernel.

Quillen also developed a theory of derived functors. Here he ran into the same problem Grothendieck and Verdier did; it is unreasonable to expect a functor between model categories to preserve weak equivalences. However, in a model category every object is equivalent to a cofibrant object, and also to a fibrant object. So for a functor F between model categories to have a left (resp. right) derived functor, it is enough for it to preserve weak equivalences between cofibrant (resp. fibrant) objects. In the ensuing 40 years since [Qui67], experience has shown that the most useful functors between model categories are left (resp. right) **Quillen functors**; these are left (resp. right) adjoints that preserve cofibrations (resp. fibrations) and trivial cofibrations (resp. trivial fibrations). Quillen also introduced what we now call a **Quillen equivalence**, which is a Quillen functor whose derived functor is an equivalence of the relevant homotopy categories. He showed that Quillen equivalences preserve the extra structure in the homotopy category that he knew about—fibration and cofibration sequences and Toda brackets—but he knew that there should be more structure that is preserved as well.

Finally, Quillen wanted to define a notion of homology that would work in a general model category, and would recover singular homology for topological spaces and the usual homology for chain complexes. Here he had the brilliant idea of considering the abelian group objects $\mathcal{C}_{\mathrm{ab}}$ in a model category $\mathcal{C}$. Quillen needed to assume that there is an abelianization functor $\mathcal{C} \to \mathcal{C}_{\mathrm{ab}}$ left adjoint to the inclusion functor, that $\mathcal{C}_{\mathrm{ab}}$ inherits a model structure from this adjunction, and that model structure is what we now call **stable**. Under these conditions, the left derived functor of abelianization is what we now call the **Quillen homology**. In the special case of simplicial commutative rings, this is **Andre-Quillen homology**. Note that Quillen homology $QH(X)$ is an abelian group object; to get actual cohomology groups we need to choose another abelian group object A and take maps in the homotopy category of abelian group objects from $QH(X)$ to A.

3. After Quillen

We now describe some of the work on model categories and the uses of them since Quillen's seminal paper. We begin with the structural theory of model categories and then describe two important applications of model categories.

Some small changes have been made in Quillen's axioms; one generally assumes now that a model category has all small limits and colimits, whereas Quillen only assumed closure under finite limits and colimits. Quillen also originally distinguished between model categories and closed model categories, whereas now everyone uses closed model categories, so that the adjective "closed" is usually dropped. At the time of [Qui67], Ken Brown's lemma [Bro73], which asserts that any functor between model categories that preserves trivial cofibrations preserves all weak equivalences between cofibrant objects, had not been proved yet. This lemma is the basis for the definition of Quillen functors given above.

On a more fundamental level, Quillen realized that he did not have a complete definition of the homotopy theory associated to a model category. He knew about cofibration and fibration sequences and about Toda brackets, but he knew there should be more structure. We now know that the homotopy category of a model category is naturally enriched, tensored, and cotensored over the homotopy category of simplicial sets, so that there is a space of maps between any two objects in a model category. The path components of this space of maps form the maps in the homotopy category, but of course there are higher homotopy groups as well and these encode homotopies between homotopies. The key developments here were Dwyer and Kan's theory of hammock localization [DK80] and Reedy's theory of diagrams in a model category [Ree74].

In addition, the theory of homotopy limits and colimits had not been developed yet. Given a model category $\mathcal{C}$ and a small category $\mathcal{I}$, one would expect the category $\mathcal{C}^{\mathcal{I}}$ of $\mathcal{I}$-diagrams in $\mathcal{C}$ to again form a model category, and the colimit and limit functors to have left and right derived functors, respectively. These are the homotopy colimit and limit functors, whose domain is the homotopy category of diagrams, not the category of diagrams in the homotopy category. There are some technical difficulties with this construction, as the category of diagrams is in fact not always a model category, explaining the many different approaches taken in the literature. Homotopy colimits and limits were first introduced for simplicial sets by Bousfield and Kan [BK72]; they are described in general in [Hir03] and [DHKS04]. Another solution to the homotopy colimit and limit problem was developed by Thomason, who changed the definition of model category slightly so that the category $\mathcal{C}^{\mathcal{I}}$ is more often a Thomason model category (see [Wei01]).

Nowadays, we think of the homotopy category of a model category as being a quasi-category or an $(\infty, 1)$-category in the sense of Boardman-Vogt [BV73], Joyal [Joy02] and Lurie [Lur09]. The quasi-category retains all of the homotopical information of the model category, but it is still less rigid. One way to put this is that theoretical constructions are easier to do in quasi-categories but computations almost always require a model category. As an example, diagrams in a quasi-category always form another quasi-category, and this can be used to define homotopy limits and colimits in quasi-categories. For diagrams in a model category to form another model category, technical hypotheses are needed. However, to compute a specific homotopy limit, it is almost always better to replace a diagram by a tower of fibrations in an appropriate model category structure.

Two of the most important applications of model categories have been the construction of a closed symmetric monoidal model category whose homotopy category is the stable homotopy category and Voevodsky's creation of the motivic stable homotopy category. In both of these, the main benefit (in the author's view) of the notion of model category was to focus attention on constructing the correct category, confident that the theory of model categories would handle the homotopy theory. We describe this for the theory of symmetric spectra [HSS00], but a similar story could be told for the S-modules of [EKMM97] and the later theory of orthogonal spectra [MMSS01]. A *spectrum* in the classical sense is a sequence of pointed spaces X_n together with maps $\epsilon_n \colon \Sigma X_n \to X_{n+1}$ from the suspension of one space to the next. Spectra form a model category [BF78], and the homotopy category is the standard stable homotopy category of [Vog70] and [Ada74]. Spectra are clearly based on the symmetric monoidal category of sequences of spaces X_n, where

$$(X \otimes Y)_n = \bigvee_{j+k=n} X_j \wedge Y_k.$$

In fact, they are S-modules, where S is the sequence of spheres $S^0, S^1, S^2, \dots$, where our model for S^n is $(S^1)^{\wedge n}$. But this is a noncommutative monoid in the category of sequences because the twist map on $S^1 \wedge S^1$ induces multiplication by -1 on homology, so is not the identity. This led Jeff Smith to introduce the category of symmetric sequences X_n, in which X_n has an action of the symmetric group Σ_n and the twist isomorphism involves shuffle permutations. In this case, the spheres again form a monoid S (with the Σ_n action on $(S^1)^{\wedge n}$ coming from permuting the coordinates), but now that monoid is commutative. The category of S-modules, usually called *symmetric spectra*, is then a symmetric monoidal category and it is in fact a model category whose homotopy category is equivalent as a symmetric monoidal category to the stable homotopy category [HSS00]. For this to work the theory of symmetric monoidal model categories had to be developed, a key step of which was the paper [SS00]. In addition, the model structure on symmetric spectra is subtle, as the weak equivalences are not the stable homotopy isomorphisms, and most objects are neither cofibrant nor fibrant.

The motivic story of Morel and Voevodsky [MV99] has similar features. The first step is to enlarge the category of schemes so it contains colimits. This is accomplished by taking simplicial sheaves in the Nisnevich topology. One should think of this as adding colimits to schemes (which are representable sheaves) while preserving the good colimits you already have. The homotopy theory is then interesting because we have two circles. On the one hand, there is the simplicial circle S^1, which has nothing to do with schemes. On the other hand, there is the affine circle $\mathbf{A}^1 - \{0\}$. From the model category point of view, the key step is making the weak equivalences be the maps which induce isomorphisms on the resulting bigraded homotopy groups. One can then stabilize using a theory analogous to spectra or symmetric spectra [Hov01] . Voevodsky used this theory and related ideas to resolve the Milnor conjecture [Voe03].

References

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics. MR 53 #6534

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of γ-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Springer, Berlin, 1978, pp. 80–130. MR 80e:55021

[BK72] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 304. MR 51 #1825

[Bro73] Kenneth S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458. MR 0341469 (49 #6220)

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin, 1973. MR 0420609 (54 #8623a)

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR 2102294 (2005k:18027)

[DK80] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440. MR 584566 (81m:55018)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR 97h:55006

[EZ50] Samuel Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology*, Ann. of Math. (2) **51** (1950), 499–513. MR 0035434 (11,734e)

[GJ99] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR 2001d:55012

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR 0210125 (35 #1019)

[Har66] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 0222093 (36 #5145)

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR 2003j:18018

[Hov99] Mark Hovey, *Model categories*, American Mathematical Society, Providence, RI, 1999. MR 99h:55031

[Hov01] ———, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), no. 1, 63–127. MR 1 860 878

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. MR 2000h:55016

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly. MR 1935979 (2003h:55026)

[Kan56] Daniel M. Kan, *Abstract homotopy. III*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 419–421. MR 0087937 (19,440a)

[Kan58] ———, *Adjoint functors*, Trans. Amer. Math. Soc. **87** (1958), 294–329. MR 0131451 (24 #A1301)

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[May92] J. Peter May, *Simplicial objects in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992, Reprint of the 1967 original, also available at `http://www.math.uchicago.edu/~may/`. MR 1206474 (93m:55025)

[Mil57] John Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. (2) **65** (1957), 357–362. MR 0084138 (18,815d)

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR 2001k:55025

[Moo58] John C. Moore, *Semi-simplicial complexes and Postnikov systems*, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 232–247. MR 0111027 (22 #1894)

[MV99] Fabien Morel and Vladimir Voevodsky, $\mathbf{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 90, 45–143 (2001). MR 1813224 (2002f:14029)

[Qui64] Daniel Quillen, *Formal properties of over-determined systems of linear partial differential equations*, Ph.D. thesis, Harvard University, 1964.

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR 36 #6480

[Ree74] Christopher Leonard Reedy, *Homology of algebraic theories*, ProQuest LLC, Ann Arbor, MI, 1974, Thesis (Ph.D.)–University of California, San Diego. MR 2624278

[SS00] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511. MR 1734325 (2001c:18006)

[Voe03] Vladimir Voevodsky, *Motivic cohomology with* $\mathbf{Z}/2$*-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, 59–104. MR 2031199 (2005b:14038b)

[Vog70] Rainer Vogt, *Boardman's stable homotopy category*, Matematisk Institut, Aarhus Universitet, Aarhus, 1970, Lecture Notes Series, No. 21. MR 43 # 1187

[Wei01] Charles Weibel, *Homotopy ends and Thomason model categories*, Selecta Math. (N.S.) **7** (2001), no. 4, 533–564. MR 1872081 (2002m:18011)

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