OPERATIONS AND CO-OPERATIONS IN MORAVA E-THEORY

MARK HOVEY

Abstract. Let $E = E_n$ denote the Morava E-theory spectrum, and let Γ be the Morava stabilizer group of ring spectrum isomorphisms of E. We revisit the folklore isomorphism $\pi_* L_{K(n)}(E \wedge E) \cong C(\Gamma, E_*)$ of graded formal Hopf algebroids, and its dual isomorphism $E^*E \cong E_*[[\Gamma]]$.

Contents

Introduction

Let p denote an integer prime, n denote a nonnegative integer, and let E denote the Morava E-theory spectrum $E_{p,n}$. This is the Landweber exact ring spectrum with

$$E_* \cong W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}],$$

where the degree of u_i is 0 for all i and the degree of u is 2. There is an algebra map $BP_* \to E_*$ that takes v_i to $u_i u^{p^i - 1}$ for $i < n$, takes v_n to $u^{p^n - 1}$, and takes v_i to 0 for $i > n$.

Date: January 5, 2004.

The Morava E-theory spectrum E is very important in algebraic topology. It is local with respect to Morava K-theory $K(n)$, and plays a major role in the structure of the $K(n)$-local homotopy category [HS99]. Let Γ denote the group of ring spectrum automorphisms (in the stable homotopy category) of E. Then Γ is a version of the Morava stabilizer group [Str00, Proposition 4], and the famous result of Hopkins-Miller [Rez98] says that E is in fact an A_∞-ring spectrum and that Γ is isomorphic to the group of components of the space of A_∞ self-maps of E.

It is then natural to compute the operations E^*E and co-operations E_*E of E. Because E is $K(n)$-local, it turns out to be more natural to look at $E^\vee_*E = \pi_*L_{K(n)}(E \wedge E)$ rather than the actual co-operations. The answer has then been known to the experts for quite some time, though Morava did not quite state it in his original paper [Mor85]; $E^\vee_*E$ is isomorphic to $C(\Gamma, E_*)$, the ring of continuous functions from the profinite group Γ to E_* with its $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$ is the unique homogeneous maximal ideal. Also, E^*E is isomorphic to the completed twisted group ring $E_*[[\Gamma]]$.

Various proofs of these results have appeared in the literature over the years. Baker seems to have given the first proof of the co-operation result in [Bak95]; see also [Bak89]. A very short treatment is given in [DH04, Proposition 2.2], and a spectrum version of the isomorphism $E^*E \cong E_*[[\Gamma]]$ appears in Proposition 2.4 of [GHMR03]. To this author's eye, at least, these proofs all do not give the calculation the respect it deserves. That is, the general assumption is that Morava essentially proved the result in [Mor85], and it is just a matter of getting efficiently from what Morava states to the actual result. Thus, many details are left out and this author, for one, finds the proofs difficult to follow. Strickland approaches the result from a new perspective in [Str00, Theorem 12]. But the calculation of $E^\vee_*E$ is a side issue in Strickland's paper, so he too leaves out many details.

In this paper, the author has tried to present the calculation of $E^\vee_*E$ in a fairly self-contained and conceptual way. Essentially, he has written a proof that he himself can understand, in the hope that this will also be useful to others. There are, after all, a great many details to be worked out. The basic idea, following Morava, is that $E^\vee_*E$ and $C(\Gamma, E_*)$ should represent the same functor. However, it is not at all obvious that this is the case. The first reduction is to divide by the maximal ideal $\mathfrak{m}$, and to eliminate the grading, reducing us to showing that K_0E and $C(\Gamma, \mathbb{F}_{p^n})$ represent the same functor. Even after identifying Γ as a profinite topological group with an appropriate version of the Morava stabilizer group, which itself requires some nontrivial theory of Landweber exact spectra and profinite groups, this is still not clear. What is fairly simple to check is that K_0E and $C(\Gamma, \mathbb{F}_{p^n})$ have the same $\overline{\mathbb{F}_p}$-valued points, where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$. The we use the fact that both algebras are ind-étale over $\mathbb{F}_{p^n}$ to complete the proof.

Here is an outline of the paper. Following Strickland [Str00], we first define the map

$$\Phi\colon E^\vee_*E \to C(\Gamma, E_*)$$

in Section 1. We also show that Γ is profinite and acts continuously on E_*. We then prove, again following Strickland's proof, that both $E^\vee_*E$ and $C(\Gamma, E_*)$ are pro-free E_*-modules in Section 2. It follows that to prove Φ is an isomorphism, it suffices to prove that the map

$$\Psi\colon K_0E \to C(\Gamma, \mathbb{F}_{p^n})$$

is an isomorphism, where $K = E/\mathfrak{m}$ is a variant on Morava K-theory. At this point, our proof diverges from Strickland's, as we now introduce the notion of ind-étale algebras in Section 3. We prove that both $K_0 E$ and $C(\Gamma, \mathbb{F}_{p^n})$ are ind-étale $\mathbb{F}_{p^n}$-algebras. It follows that Ψ is an isomorphism if and only if $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism, where $\overline{\mathbb{F}_p}$ denotes the algebraic closure of $\mathbb{F}_p$. We then prove that $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism in Section 4, completing the proof that Φ is an isomorphism. This requires identification of the group Γ, also done by Strickland in [Str00] but with many details left out.

Having completed the proof that Φ is an isomorphism in Theorem 4.11, we turn to cohomology operations in Section 5. We define the twisted completed group ring $E_*[[\Gamma]]$ and show that the natural map

$$E_*[[\Gamma]] \to E^* E$$

induced by the inclusion of Γ in $E^0 E$ is an isomorphism. We turn to Hopf algebroid structure in Section 6, defining the notion of a graded formal Hopf algebroid, showing that both $E_*^\vee E$ and $C(\Gamma, E_*)$ carry such structure, and proving that Φ preserves it. We close the paper with an appendix where we show that the topology on Γ is entirely determined by its group structure. This is also proved in [Str00], but again with details left out.

The author would like to thank Neil Strickland, whose paper [Str00] was the inspiration for this one. It was also Strickland who first suggested the word "étale" to the author, which turned out to be the key idea to making the proof more conceptual. He would also like to thank Daniel Davis for his careful reading of the paper and his encouragement.

1. The map Φ

In this section we define the map

$$\Phi\colon\ E_*^\vee E \to C(\Gamma, E_*).$$

We will frequently use the notation $X \widetilde{\wedge} Y$ to mean $L_{K(n)}(X \wedge Y)$. With this notation, $E_*^\vee X = \pi_*(E \widetilde{\wedge} X)$.

To construct Φ, following Strickland [Str00, Theorem 12], we recall that there is a natural topology on $[X, Y]$, for X, Y $K(n)$-local spectra [HS99, Section 11] (or objects in any algebraic stable homotopy category [HPS97, Section 4.4]). Given any F that is small in the $K(n)$-local category, and any map $h\colon F \to X$, we define U_h to be the set of all $f \in [X, Y]$ with $fh = 0$. The U_f define a basis of neighborhoods of 0 in $[X, Y]$, and then $V \subseteq [X, Y]$ is open if and only if for all $v \in V$, there is an h such that $v + U_h \subseteq V$. Equivalently, V is open if and only if for all v in V there is an h such that $fh = vh$ implies $f \in V$.

There is a map

$$\sigma\colon\ \Gamma \times E_*^\vee E \to E_*$$

defined by letting $\sigma(\gamma, a)$ be the composite

$$S^m \xrightarrow{a} E \widetilde{\wedge} E \xrightarrow{1 \widetilde{\wedge} \gamma} E \widetilde{\wedge} E \xrightarrow{\mu} E.$$

As pointed out by Strickland, if we give $E_*^\vee E$, E_*, and $\Gamma \subseteq [E, E]$ their natural topologies as sets of maps in the $K(n)$-local category [HS99, Section 11], the map

σ is continuous by Proposition 11.3 and Proposition 11.1 of [HS99]. Its adjoint is therefore a map

$$\Phi\colon\ E^{\vee}_{*}E \to C(\Gamma, E_{*}),$$

where continuity is determined by the natural topologies. It is a straightforward diagram chase to verify that Φ is a map of E_*-modules. Using the fact that Γ consists of ring spectrum automorphisms, one can also check by a diagram chase that Φ is a map of E_*-algebras.

The natural topology on E_* is the $\mathfrak{m}$-adic topology, by the method of [HS99, Proposition 11.9]. Note that the E used in [HS99] is not the same E we use here. The same methods apply to both versions of E, however.

We will of course need to determine the topological group Γ. For the moment, however, we content ourselves with some basic structural results.

Proposition 1.1. *The set $[E, E]$ is a first countable compact Hausdorff topological ring in the natural topology, and the topology is defined by a set of open left ideals. Furthermore, the action of $[E, E]$ on $[X, E]$ is continuous for all $K(n)$-local spectra X.*

Proof. The group $[X, Y]$ is always a topological group in the natural topology. Since the multiplication in $[E, E]$ is given by composition, it is continuous by [HS99, Proposition 11.1(a)], and so $[E, E]$ is a topological ring. The same argument shows that the action of $[E, E]$ on $[X, E]$ is continuous. Now suppose F is small in the $K(n)$-local category. The argument of [HS99, Theorem 8.6] shows that E^*F is finite in each degree. Then [HS99, Proposition 11.5] implies that $[E, E]$ is compact Hausdorff. To see that $[E, E]$ is first countable, note that there are only countably many (isomorphism classes of) small objects in the $K(n)$-local category by the results of [HS99, Section 8]. Given a small object F, we have just seen that $[F, E]$ is finite. Hence there are only countably many basic neighborhoods U_h of 0, so $[E, E]$ is first countable. It is clear that the U_h are left ideals. □

Corollary 1.2. *The group Γ is a first countable Hausdorff topological group whose topology is defined by a set of open subgroups. Furthermore, the action of Γ on E_* is continuous when E_* is given the $\mathfrak{m}$-adic topology.*

Proof. In general, if A is a topological ring defined by a set of open left ideals, then $A^{\times}$ is a topological group defined by a set of open subgroups. Indeed, if U is an open left ideal in A, then one can check that $V = (e+U) \cap A^{\times}$ is an open subgroup in $A^{\times}$. If $\{U_i\}$ defines the topology on A, then the corresponding set $\{V_i\}$ defines the subspace topology on $A^{\times}$. It is clear that the multiplication map is continuous; to see that the inverse map is continuous, note that $(x(x^{-1}Vx))^{-1} = x^{-1}V$. Our group Γ is a subgroup of $[E, E]^{\times}$, so is also a first countable Hausdorff topological group whose topology is defined by a set of open subgroups.

Proposition 1.1 shows that the action of Γ on E_* is continuous when E_* is given the natural topology, and we have seen already that the natural topology on E_* is the $\mathfrak{m}$-adic topology. □

We want to conclude that Γ is profinite. For this, we need to know that Γ is compact, according to [DdSMS99, Definition 1.1]. Since $[E, E]$ is compact, it suffices to show that Γ is closed in $[E, E]$.

Lemma 1.3. *Suppose A is a compact Hausdorff topological ring. Then $A^{\times}$ is closed in A.*

Proof. Note that $A^\times = B \cap C$, where B is the set of all elements with a right inverse and C is the set of all elements with a left inverse. We will show that B is closed; a similar proof shows that C is closed as well. Let $\mu\colon A \times A \to A$ denote the multiplication map, and let D denote $\mu^{-1}(1)$, which is closed since A is Hausdorff. Then $B = \pi_1 D$, but π_1 is a map from the compact space $A \times A$ to the Hausdorff space A, so is a closed mapping. ☐

Theorem 1.4. *The group Γ is a first countable profinite group.*

Proof. In view of Corollary 1.2 and Lemma 1.3, it suffices to show that the set S of ring spectrum maps in $[E, E]$ is closed. Note that $S = S_1 \cap S_2$, where S_1 is the set of maps compatible with the multiplication μ and S_2 is the set of maps compatible with the unit η. Certainly S_2 is closed, since it is the inverse image of η under the continuous composition

$$[E, E] \to [S, E] \times [E, E] \xrightarrow{\circ} [S, E].$$

Here the first map sends f to (η, f), and the second map is composition. On the other hand, S_1 is the equalizer of the maps

$$[E, E] \xrightarrow{\mu^*} [E \wedge E, E]$$

and

$$[E, E] \xrightarrow{\Delta} [E, E] \times [E, E] \xrightarrow{\wedge} [E \wedge E, E \wedge E] \xrightarrow{\mu_*} [E \wedge E, E].$$

The maps μ^* and μ_* are continuous since they are just composition, and the map $\wedge$ is continuous by [HS99, Proposition 11.3]. Proposition 11.5 of [HS99] tells us that $[E \wedge E, E]$ is Hausdorff, and the equalizer of two continuous maps into a Hausdorff space is always closed. ☐

2. Pro-free modules

In this section, we show that both $E^\vee_* E$ and $C(\Gamma, E_*)$ are pro-free E_*-modules. This allows us to conclude that Φ is an isomorphism if and only if the induced map

$$\Psi\colon\ K_0 E \to C(\Gamma, \mathbb{F}_{p^n})$$

is an isomorphism.

Recall that, if R is a complete local ring, an R-module is called **pro-free** if it is the completion of a free module. Our particular ring E_* is graded, so when we take the completion of a graded module we must do so in the graded sense. However, all our E_*-modules will be evenly graded, and since E_* has a unit in degree 2, there is an equivalence of categories between evenly graded E_*-modules and E_0-modules. This equivalence takes an evenly graded E_*-module M to M_0, and an E_0-module N to $N[u, u^{-1}]$, with N in degree 0.

We need to show that the E_*-modules we deal with are pro-free. For this, we recall the spectrum $K = E/I_n$. This spectrum can be made into a ring spectrum by using, for example, the theory of bordism with singularities or its modern replacement [EKMM97, Chapter V]. It is in fact a field spectrum, and is additively isomorphic to a wedge of suspensions of $K(n)$. Note that $K_* \cong \mathbb{F}_{p^n}[u, u^{-1}]$.

Proposition 2.1. *The E_*-module $E^\vee_* E$ is pro-free and concentrated in even dimensions.*

Proof. We use the method of Proposition 8.4(f) of [HS99], which implies that it suffices to check that K_*E is concentrated in even degrees. Landweber exactness of E implies that $K_*E \cong K_*BP \otimes_{BP_*} E_*$, but the Atiyah-Hirzebruch spectral sequence implies that $K_*BP \cong K_*[t_1, t_2, \dots]$ with $|t_i| = 2(p^i - 1)$ as in [Rav86, Lemma 4.1.7]. □

To show that $C(\Gamma, E_*)$ is pro-free will require more work. We begin by analyzing $C(G, R)$ when G is profinite and R is discrete.

Lemma 2.2. *Suppose $G = \lim G/U$ is a profinite group and R is a discrete commutative ring. Then the natural map of rings $\rho\colon \operatorname{colim}_U F(G/U, R) \to C(G, R)$ is an isomorphism, where U runs through the open normal subgroups of G,*

This lemma works when R is graded as well.

Proof. The map ρ sends a map $G/U \to R$ to the composite $G \to G/U \to R$. Since the reduction maps are surjective, ρ is injective. Now suppose $f\colon G \to R$ is continuous. For each $x \in G$, choose an open normal subgroup N_x such that $f(xN_x) = f(x)$. The xN_x form an open cover of G, so there is a finite subcover $\{x_1N_{x_1}, \dots, x_kN_{x_k}\}$. Let $N = N_{x_1} \cap \dots \cap N_{x_k}$. We claim that f is induced by a function $G/N \to \mathbb{Z}$. Indeed, suppose x and y are congruent modulo N. Now x must lie in some $x_iN_{x_i}$, so x is congruent to x_i modulo N_{x_i}. It follows that y is also congruent to x_i modulo N_{x_i}, and so $f(x) = f(x_i) = f(y)$, as required. □

In fact, the dependence of $C(G, R)$ on R is very simple.

Proposition 2.3. *Suppose G is a profinite group and R is a discrete commutative ring. Then the natural map $\sigma\colon R \otimes C(G, \mathbb{Z}) \to C(G, R)$ defined by $\sigma(r, f)(g) = f(g)r$ is an isomorphism.*

Again, this proposition holds if R is graded as well, where $C(G, \mathbb{Z})$ is thought of as a graded ring concentrated in degree 0.

Proof. When G is finite, the proposition is clear since both sides are free R-modules of rank $|G|$, and σ takes the basis $\{1 \otimes g^*\}$ to the basis $\{g^*\}$, where g^* is the function that takes g to 1 and everything else to 0. When $G = \lim G/U$ is profinite, we take the direct limits of the isomorphisms for G/U and use Lemma 2.2. □

We need a version of Proposition 2.3 when R is complete in the $\mathfrak{a}$-adic topology, for some ideal $\mathfrak{a}$. In this case, if S is a discrete ring such as $C(G, \mathbb{Z})$, we define $R \widehat{\otimes} S$ to be the completion of $R \otimes S$ with respect to the image of $\mathfrak{a} \otimes S$ in $R \otimes S$.

Proposition 2.4. *Suppose G is a profinite group and R is a commutative ring that is complete in the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a}$. Then there is a natural isomorphism*

$$R \widehat{\otimes} C(G, \mathbb{Z}) \xrightarrow{\cong} C(G, R).$$

As usual, this isomorphism will work in the graded case as well, as long as $\mathfrak{a}$ is homogeneous.

Proof. Take the inverse limit of the isomorphisms

$$\sigma\colon R/\mathfrak{a}^i \otimes C(G, \mathbb{Z}) \to C(G, R/\mathfrak{a}^i),$$

of Proposition 2.3. □

We can now prove that $C(\Gamma, E_*)$ is pro-free.

Theorem 2.5. *Suppose G is a first countable profinite group and R is a commutative ring that is complete in the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a}$. Then $C(G, R)$ is pro-free as an R-module.*

This theorem holds in either the graded or ungraded case.

Proof. Proposition 2.4 tells us that $C(G, R) \cong R\widehat{\otimes}C(G, \mathbb{Z})$. Hence it suffices to show that $C(G, \mathbb{Z})$ is a free abelian group. But Lemma 2.2 tells us that $C(G, \mathbb{Z}) \cong \operatorname{colim}_U F(G/U, \mathbb{Z})$. Since G is first countable, we can make this colimit run over a chain $\cdots \subseteq U_k \subseteq \cdots \subseteq U_0$. Each group $F(G/U_k, \mathbb{Z})$ is a finitely generated free abelian group, and each map $F(G/U_k, \mathbb{Z}) \to F(G/U_{k+1}, \mathbb{Z})$ is a split monomorphism. Hence $C(G, \mathbb{Z})$ is a free abelian group. □

We then have the following general lemma about maps of pro-free modules.

Lemma 2.6. *Suppose R is a Noetherian regular complete local ring with maximal ideal $\mathfrak{m}$. A map $f\colon M \to N$ of pro-free modules is an isomorphism if and only $R/\mathfrak{m} \otimes_R f$ is an isomorphism.*

We are of course interested in this lemma when R is graded, but the proof is the same in any case.

Proof. It is proved in [HS99, Proposition A.13] that if $R/\mathfrak{m}\otimes_R f$ is a monomorphism, then f is a split monomorphism. The cokernel C of f is then a summand in the complete module N, but $C/\mathfrak{m}C = 0$. It follows that $C = 0$. □

We then have the following theorem.

Theorem 2.7. *The map $\Phi\colon E^\vee_* E \to C(\Gamma, E_*)$ is an isomorphism if and only if the map $\Psi\colon K_0E \to C(\Gamma, \mathbb{F}_{p^n})$ induced by Φ is an isomorphism.*

Proof. Since $E^\vee_* E$ and $C(\Gamma, E_*)$ are pro-free by Proposition 2.1 and Theorem 2.5, Lemma 2.6 implies that Φ is an isomorphism if and only $\Phi/\mathfrak{m}$ is an isomorphism. The method of [HS99, Proposition 8.4(e)] implies that $E^\vee_* E/\mathfrak{m} \cong K_*E$. Proposition 2.4 and Proposition 2.3 imply that

$$\begin{aligned} C(\Gamma, E_*)/\mathfrak{m} &\cong (E_*\widehat{\otimes}C(\Gamma, \mathbb{Z}))/\mathfrak{m} \\ &\cong (E_* \otimes C(\Gamma, \mathbb{Z}))/\mathfrak{m} \cong (E_*/\mathfrak{m}) \otimes C(\Gamma, \mathbb{Z}) \\ &\cong C(\Gamma, K_*). \end{aligned}$$

Now K_*E is concentrated in even dimensions by Proposition 2.1, as is $C(\Gamma, K_*)$. Since both contain the unit u in degree 2, $\Phi/\mathfrak{m}$ is an isomorphism if and only if the degree 0 part Ψ of $\Phi/\mathfrak{m}$ is an isomorphism. □

3. Ind-étale algebras

The object of this section is to show that

$$\Psi\colon K_0E \to C(\Gamma, \mathbb{F}_{p^n})$$

is an isomorphism if and only if $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism of sets, where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$, and $\mathbf{Alg}_{\mathbb{F}_{p^n}}$ is the category of $\mathbb{F}_{p^n}$-algebras. We will accomplish this by showing that both K_0E and $C(\Gamma, \mathbb{F}_{p^n})$ are ind-étale $\mathbb{F}_{p^n}$-algebras, and proving a general result about maps between such algebras.

3.1. **Ind-étale algebras.** We first develop the algebraic theory of ind-étale algebras. For a general commutative ring k, a finitely presented k-algebra is called **étale** if, whenever N is a square 0 ideal in a k-algebra C, the natural map

$$\mathbf{Alg}_k(A, C) \to \mathbf{Alg}_k(A, C/N)$$

is an isomorphism. A is called **smooth** if this map is always surjective, and **unramified** if it is always injective. These definitions are taken from [DG70, Section I.4.3], where of course one works with arbitrary schemes.

If k is a field, the category of étale k-schemes is equivalent to the category of G-sets, where G is the Galois group of the separable closure k_s over k [DG70, I.4.6.4]. Note that G is profinite, and so a G-set means a set with a continuous action of G. The equivalence takes X to the set $X(k_s)$ of its k_s-valued points. The equivalence works because if X is an étale k-scheme, then $X \otimes_k k_s$ is constant [DG70, I.4.6.2]. Now, if A is an étale algebra, then $\operatorname{Spec} A \otimes_k k_s$ is both constant and affine, so must be a finite constant scheme. Thus we have the following well-known proposition. See Proposition 18.3 and Theorem 18.4 of [KMRT98].

Proposition 3.1. *Suppose k is a field. A k-algebra A is étale if and only if $A \otimes_k k_s$ is isomorphic to a finite product of copies of k_s. Moreover, the category of étale k-algebras is anti-equivalent to the category of finite G-sets, where G denotes the Galois group of k_s/k.*

Now, the algebras we have to deal with are definitely not finitely presented, so we make the following definition.

Definition 3.2. If k is a commutative ring, we define a k-algebra A to be **ind-étale** if A is a filtered colimit of étale k-algebras.

The main reason we introduce this definition is the following lemma.

Lemma 3.3. *If k is a field and G is a profinite group, then $C(G, k)$ is an ind-étale k-algebra.*

Proof. Lemma 2.2 tells us that $C(G, k) \cong \operatorname{colim} F(G/U, k)$, where U runs through the open normal subgroups of G. Each $F(G/U, k)$ is obviously étale, since it is a product of copies of k even before tensoring with k_s. □

The terminology "ind-étale" is justified by the following proposition.

Proposition 3.4. *For a commutative ring k, the category of ind-étale k-algebras is equivalent to the category of ind-objects in the category of étale k-algebras.*

Here, given a category $\mathcal{C}$, the category of ind-objects in $\mathcal{C}$ is the category of all functors $F\colon I \to \mathcal{C}$, where I can be any filtered small category. The morphisms in this category from F to G are defined to be $\lim_\alpha \operatorname{colim}_\beta \mathcal{C}(F(\alpha), G(\beta))$. See [gro72, Section I.8.2].

Proof. The colimit is an obvious functor from ind-objects to ind-étale k-algebras. It is essentially surjective by definition. To see that it is fully faithful, we compute

$$\begin{aligned}\mathbf{Alg}_k(\operatorname{colim} A_\alpha, \operatorname{colim} B_\beta) &\cong \lim_\alpha \mathbf{Alg}_k(A_\alpha, \operatorname{colim} B_\beta) \\ &\cong \lim_\alpha \operatorname{colim}_\beta \mathbf{Alg}_k(A_\alpha, B_\beta),\end{aligned}$$

using the fact that A_α is finitely presented as a k-algebra. □

It follows from Proposition 3.1 that, if k is a field, the category of ind-étale k-algebras is anti-equivalent to the category of pro-objects in the category of finite G-sets.

Proposition 3.5. *Let G be a profinite group. The limit functor from the category of pro-objects in finite G-sets to profinite G-sets is an equivalence of categories.*

The morphisms in the category of profinite G-sets are continuous equivariant maps from X to Y, denoted $C_G(X,Y)$.

Proof. The limit functor is essentially surjective by definition. To determine whether it is fully faithful, we compute

$$C_G(\lim X_\alpha, \lim Y_\beta) \cong \lim_\beta C_G(\lim X_\alpha, Y_\beta).$$

There is a canonical map

$$\sigma\colon\ \operatorname{colim}_\alpha C_G(X_\alpha, Y_\beta) \to C_G(\lim X_\alpha, Y_\beta).$$

We claim that this map is an isomorphism. To see that σ is surjective, note that a continuous G-map $f\colon\ \lim X_\alpha \to Y_\beta$ is determined by a partition of $\lim X_\alpha$ into open and closed sets $U_y = f^{-1}(y)$ for $y \in Y_\beta$, with $gU_y \subseteq U_{gy}$. A basis for the topology on $\lim X_\alpha$ is given by the sets $\pi_\gamma^{-1}(V_\gamma)$, where $\pi_\gamma\colon\ \lim X_\alpha \to X_\gamma$ is the evident map and V_γ is an arbitrary subset of X_γ. Because our indexing category is filtered, this basis is closed under finite intersections and finite unions. Since $\lim X_\alpha$ is compact, this means that a set that is both open and closed must be of the form $\pi_\gamma^{-1}(V_\gamma)$. Again using the fact that the indexing category is filtered, we conclude that there is a γ and subsets V_y in X_γ such that $U_y = \pi_\gamma^{-1}(V_y)$ for all $y \in Y_\beta$. It follows that f is in the image of $C_G(X_\gamma, Y_\beta)$, and hence that σ is surjective. This implies that the inverse limit functor is full.

To see that σ is injective, we use an argument we learned from Neil Strickland. Suppose we have two maps $f\colon\ X_\alpha \to Y_\beta$ and $f'\colon\ X_{\alpha'} \to Y_\beta$ with $f \circ \pi_\alpha = f' \circ \pi_{\alpha'}$. Since the diagram $\{X_\gamma\}$ is filtered, we can assume $\alpha = \alpha'$. Choose an $x \in X_\alpha$ with $f(x) \neq f'(x)$. Let $\mathcal{I}$ denote the indexing category of the pro-object X, and let $\mathcal{C}$ denote the category of all pairs (γ, i) where $i\colon\ \gamma \to \alpha$ is a morphism in $\mathcal{I}$. Then $\mathcal{C}$ is itself a filtered category, and we can define a functor from $\mathcal{C}$ to finite sets by taking (γ, i) to $(X(i))^{-1}(x)$. The inverse limit of this functor is $\pi_\alpha^{-1}(x)$, which must be empty since $f \circ \pi_\alpha = f' \circ \pi_\alpha$. By [DdSMS99, Proposition 1.1.4], it follows that there exists an $i\colon\ \gamma \to \alpha$ such that $(X(i))^{-1}(x) = \emptyset$. Since there are only finitely many elements x and $\mathcal{I}$ is filtered, it follows that there exists a morphism $j\colon\ \delta \to \alpha$ such that $(X(j))^{-1}(x)$ is empty for all x with $f(x) \neq f'(x)$. Hence $f \circ X(j) = f' \circ X(j)$, and so f and f' represent the same element of $\operatorname{colim}_\alpha C_G(X_\alpha, Y_\beta)$. □

We reach the following conclusion.

Theorem 3.6. *Suppose k is a field with separable closure k_s. Let G be the Galois group of k_s over k. The functor that takes a k-algebra A to $\mathbf{Alg}_k(A, k_s)$ defines an anti-equivalence of categories between ind-étale k-algebras and profinite G-sets. In particular, if $f\colon\ A \to B$ is a map of ind-étale k-algebras, then f is an isomorphism if and only if the induced map*

$$\mathbf{Alg}_k(B, k_s) \to \mathbf{Alg}_k(A, k_s)$$

is an isomorphism.

3.2. **Identification of K_0E.** We have seen already in Lemma 3.3 that $C(\Gamma, \mathbb{F}_{p^n})$ is an ind-étale $\mathbb{F}_{p^n}$-algebra. To show that K_0E is an ind-étale $\mathbb{F}_{p^n}$-algebra will require considerably more work. The goal of this section is to show that

$$K_0E \cong \mathbb{F}_{p^n} \otimes_V VT[t_0, t_0^{-1}] \otimes_V \mathbb{F}_{p^n}$$

where V is BP_* thought of as an ungraded ring, and VT is BP_*BP thought of as an ungraded ring. Thus $(V, VT[t_0, t_0^{-1}])$ is the Hopf algebroid that represents the groupoid of p-typical formal group laws and arbitrary isomorphisms between them (see [Rav86, p. 365]).

We begin by defining $R_* = BP_*[u, u^{-1}]$, where u has degree 2. Then R_* is a free BP_*-algebra, so is obviously Landweber exact. Hence we get a commutative ring spectrum R, and an isomorphism $R_*X \cong R_* \otimes_{BP_*} BP_*X$ [HS99, Section 2]. In particular, $R_*R \cong R_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} R_*$.

Lemma 3.7. *There is an isomorphism of Hopf algebroids*

$$\tau\colon\ (V[u, u^{-1}], VT[t_0, t_0^{-1}][u, u^{-1}]) \cong (R_*, R_*R).$$

Proof. A map of graded rings from R_* to S is a pair $(F(x, y), u)$ where $F(x, y)$ is a homogeneous p-typical formal group law over S and u is a unit in S_2. To say that $F(x, y) = \sum_{i,j} a_{ij}x^iy^j$ is homogeneous means that each a_{ij} is homogeneous of degree $2(i + j - 1)$. On the other hand, a map of graded rings from $V[u, u^{-1}]$ to S is a pair $(F_0(x, y), y)$ where F_0 is p-typical formal group law over S_0 and u is a unit in S_2. Given $(F(x, y), u)$ we define

$$F_0(x, y) = \sum_{i,j} a_{ij}u^{1-i-j}x^iy^j = uF(u^{-1}x, u^{-1}y).$$

One can readily see from [Rav86, Lemma A2.1.26] that F_0 is p-typical. Conversely, given $(F_0(x, y), u)$ we define $F(x, y) = u^{-1}F_0(ux, uy)$, which is homogeneous and p-typical. This one-to-one correspondence gives us the desired isomorphism $V[u, u^{-1}] \cong R_*$.

Similarly, a map of graded rings from R_*R to S is a quintuple

$$(F(x, y), F'(x, y), u, v, \phi),$$

where $F(x, y)$ and $F'(x, y)$ are homogeneous p-typical formal group laws over S, u, v are units in S_2, and ϕ is a homogeneous strict isomorphism from F to F', in the sense that $\phi(x) = \sum b_ix^i$ with b_i homogeneous of degree $2i - 2$. On the other hand, a map of graded rings from $VT[t_0, t_0^{-1}][u, u^{-1}]$ to S is a quadruple $(F_0(x, y), F_0'(x, y), u, \psi)$, where F_0 and F_0' are formal group laws over S_0, u is a unit in degree 2, and ψ is an arbitrary isomorphism from F_0 to F_0'. Given (F, F', u, v, ϕ), we define $F_0(x, y) = uF(u^{-1}x, u^{-1}y)$ and $F_0'(x, y) = vF'(v^{-1}x, v^{-1}y)$ as above. We define $\psi(x) = v\phi(u^{-1}x)$. The reader can check that ψ is defined over S_0, and has leading term $vu^{-1}x$. We have

$$\begin{aligned} F_0'(\psi x, \psi y) &= vF'(v^{-1}\psi x, v^{-1}\psi y) = vF'(\phi(u^{-1}x), \phi(u^{-1}y)) \\ &= v\phi F(u^{-1}x, u^{-1}y) = v\phi(u^{-1}F_0(x, y)) = \psi F_0(x, y). \end{aligned}$$

Hence ψ is an isomorphism from F_0 to F_0'. Conversely, given (F_0, F_0', u, ψ). we let w be the leading coefficient $(d\psi(x)/dx)(0)$ of ψ and define

$$v = uw, F(x, y) = u^{-1}F(ux, uy), F' = v^{-1}F_0'(vx, vy), \text{ and } \phi(x) = v^{-1}\psi(ux).$$

We leave to the reader the check that $\phi(x)$ is a homogeneous strict isomorphism, and that these constructions are inverse to one another. □

We then have the following proposition.

Proposition 3.8. $E_0E \cong E_0 \otimes_V VT[t_0, t_0^{-1}] \otimes_V E_0$.

Note that E_0 is a V-algebra via the map that takes v_i to u_i for $0 \leq i < n$, takes v_n to 1, and takes v_i to 0 for $i > n$.

Proof. The map $BP_* \to E_*$ used to build $E_*(-)$ factors through R_*. Using Landweber exactness of E, we conclude that

$$E_*E \cong E_* \otimes_{R_*} R_*R \otimes_{R_*} E_*.$$

Using Lemma 3.7, we conclude that

$$\begin{aligned} E_0E[u, u^{-1}] \cong E_*E &\cong E_0[u, u^{-1}] \otimes_{V[u,u^{-1}]} VT[t_0, t_0^{-1}][u, u^{-1}] \otimes_{V[u,u^{-1}]} E_0[u, u^{-1}] \\ &\cong (E_0 \otimes_V VT[t_0, t_0^{-1}] \otimes_V E_0)[u, u^{-1}]. \end{aligned}$$

The proposition follows. □

Corollary 3.9. $K_0E \cong \mathbb{F}_{p^n} \otimes_V VT[t_0, t_0^{-1}] \otimes_V \mathbb{F}_{p^n}$.

In this corollary, the map $V \to \mathbb{F}_{p^n}$ sends all v_i to 0 except v_n, which goes to 1.

Proof. Because E is Landweber exact, E_*E is a flat E_*-module. In particular, by tensoring the sequences

$$0 \to E_*/(p, u_1, \ldots, u_{i-1}) \xrightarrow{u_i} E_*/(p, u_1, \ldots, u_{i-1}) \to E_*/(p, u_1, \ldots, u_i) \to 0$$

with E_*E, we see that $(p, u_1, \ldots, u_{n-1})$ is a regular sequence on E_*E. Hence $K_*E \cong E_*E/\mathfrak{m}$, and so $K_0E \cong E_0E/\mathfrak{m}$. Since $I_n = (p, v_1, \ldots, v_{n-1})$ is an invariant ideal in VT, the corollary follows from Proposition 3.8. □

3.3. **K_0E as an ind-étale $\mathbb{F}_{p^n}$-algebra.** Having identified K_0E, we can begin the process of proving that K_0E is an ind-étale $\mathbb{F}_{p^n}$-algebra. We begin by identifying the representing scheme for automorphisms of the Honda formal group law F_n. Recall that this is the p-typical formal group law over an $\mathbb{F}_p$-algebra S whose classifying map takes v_i to 0 for all $i \neq n$ and takes v_n to 1. Its p-series is $[p]_{F_n}(x) = x^{p^n}$.

Proposition 3.10. *Let S be an $\mathbb{F}_p$-algebra. A power series $\psi(x) \in S[[x]]$ is an automorphism of the Honda formal group law F_n if and only if* $\psi^{-1}(x) = \sum_{i=0}^{\infty}{}^{F_n} t_i x^{p^i}$ *where $t_i^{p^n} = t_i$ for all i and t_0 is a unit.*

The only if half of this proposition is proved in the proof of [Str00, Theorem 12], and the if half is stated there.

Proof. Suppose first that $\psi(x)$ is an automorphism of F_n. Since F_n is p-typical, Lemma A2.1.26 of [Rav86] implies that $\psi^{-1}(x) = \sum_{i=0}^{\infty}{}^{F_n} t_i x^{p^i}$. Since $\psi^{-1}(x)$ is also an automorphism of F_n, it commutes with the p-series. We conclude that

$$\sum_{i=0}^{\infty}{}^{F_n} t_i (x^{p^n})^{p^i} = (\psi^{-1}(x))^{p^n},$$

so
$$\sum_{i=0}^{\infty}{}^{F_n}\, t_i x^{p^{i+n}} = \sum_{i=0}^{\infty}{}^{F_n}\, t_i^{p^n} x^{p^{i+n}},$$
Hence $t_i^{p^n} = t_i$ for all i.

Conversely, suppose $\psi^{-1}(x) = \sum_{i=0}^{\infty}{}^{F_n}\, t_i x^{p^i}$, $t_i^{p^n} = t_i$ for all i, and t_0 is a unit. Define $F(x,y) = \psi F_n(\psi^{-1}x, \psi^{-1}y)$. Then F is a formal group law, and ψ is an isomorphism from F_n to F. Furthermore, F is p-typical by [Rav86, Lemma A2.1.26]. Thus $[p]_F(x) = \sum_{j=1}^{\infty}{}^{F}\, s_j x^{p^j}$ for some elements s_j of S. Note that s_j is the image of v_j under the classifying map of F, so it suffices to show that $s_j = 0$ for all j except n, and that $s_n = 1$.

We do this by comparing the p-series. On the one hand, we have
$$\psi^{-1}[p]_F(x) = \psi^{-1}(\sum_{j=1}^{\infty}{}^{F}\, s_j x^{p^j}) = \sum_{j=1}^{\infty}{}^{F_n}\, \psi^{-1}(s_j x^{p^j}) = \sum_{i=0}^{\infty}{}^{F_n} \sum_{j=1}^{\infty}{}^{F_n}\, t_i s_j^{p^i} x^{p^{i+j}}.$$
On the other hand, we have
$$\psi^{-1}[p]_F(x) = [p]_{F_n}(\psi^{-1}x) = \sum_{i=0}^{\infty}{}^{F_n}\, t_i x^{p^{i+n}},$$
using the fact that $t_i^{p^n} = t_i$. Looking at the smallest power of x that occurs, we conclude that $t_0 s_1 = 0$ (if $n > 1$), so $s_1 = 0$ since t_0 is a unit. Continuing in this fashion we see that $s_j = 0$ for $j < n$ and $s_n = 1$. Hence
$$\sum_{i=0}^{\infty}{}^{F_n}\, t_i x^{p^{i+n}} +_{F_n} \sum_{i=0}^{\infty}{}^{F_n} \sum_{j=n+1}^{\infty}{}^{F_n}\, t_i s_j^{p^i} x^{p^{i+j}} = \sum_{i=0}^{\infty}{}^{F_n}\, t_i x^{p^{i+n}}.$$
Cancelling and again recursively looking at the smallest power of x that occurs, we find that $s_j = 0$ for all $j > n$ as well. □

Corollary 3.11. *The representing ring for automorphisms of the Honda formal group law F_n is $\mathbb{F}_p[t_0, t_1, \dots]/(t_0^{p^n-1} - 1, t_1^{p^n} - t_1, \dots)$.*

We then get the following description of $K_0 E$, also obtained by Strickland in the proof of [Str00, Theorem 12].

Proposition 3.12. $K_0 E \cong \mathbb{F}_{p^n}[t_0, t_1, \dots]/(t_0^{p^n-1} - 1, t_1^{p^n} - t_1, \dots) \otimes \mathbb{F}_{p^n}$.

Proof. By Corollary 3.9, we have
$$K_0 E \cong \mathbb{F}_{p^n} \otimes_V VT[t_0, t_0^{-1}] \otimes_V \mathbb{F}_{p^n}$$
where the map $V \to \mathbb{F}_{p^n}$ takes u_i to 0 for all $i \neq n$ and takes u_n to 1. This ring represents the functor that assigns to a ring R the set of triples (r, s, ψ), where $r, s \colon \mathbb{F}_{p^n} \to R$ are ring homomorphisms and ψ is an isomorphism of formal groups from $r_* F_n$ to $s_* F_n$, where F_n is the Honda formal group law classified by the map $V \to \mathbb{F}_{p^n}$. In fact, F_n is actually defined over $\mathbb{F}_p$, so $r_* F_n = s_* F_n = F_n$. We conclude from Corollary 3.11 that
$$K_0 E \cong \mathbb{F}_{p^n} \otimes \mathbb{F}_p[t_0, t_1, \dots]/(t_0^{p^n-1} - 1, t_1^{p^n} - t_1, \dots) \otimes \mathbb{F}_{p^n},$$

which proves the proposition. □

Theorem 3.13. *K_0E is an ind-étale $\mathbb{F}_{p^n}$-algebra.*

Proof. Define

$$R'_k = \mathbb{F}_{p^n}[t_0, t_1, \ldots, t_k]/(t_0^{p^n-1} - 1, t_1^{p^n} - t_1, \ldots, t_k^{p^n} - t_k)$$

and define $R_k = R'_k \otimes \mathbb{F}_{p^n} = R'_k \otimes_{\mathbb{F}_{p^n}} (\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n})$. It is clear from Proposition 3.12 that $K_0E \cong \operatorname{colim} R_k$ as $\mathbb{F}_{p^n}$-algebras. Thus it suffices to show that R_k is an étale $\mathbb{F}_{p^n}$-algebra. Since

$$\mathbf{Alg}_{\mathbb{F}_{p^n}}(R_k, -) \cong \mathbf{Alg}_{\mathbb{F}_{p^n}}(R'_k, -) \times \mathbf{Alg}_{\mathbb{F}_{p^n}}(\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n}, -),$$

it suffices to show that both R'_k and $\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n}$ are étale. It is easy to check that $\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n}$ is an étale $\mathbb{F}_{p^n}$-algebra. Indeed, an $\mathbb{F}_{p^n}$-algebra map from $\mathbb{F}_{p^n} \otimes \mathbb{F}_{p^n}$ to A is the same thing as a ring map $\mathbb{F}_{p^n} \to A$. Now suppose N is a square-zero ideal in A. If $f, g\colon \mathbb{F}_{p^n} \to A$ become equal in A/N, then $(f(x) - g(x))^{p^n} = 0$, and so $f(x^{p^n}) = g(x^{p^n})$ for all $x \in \mathbb{F}_{p^n}$. Since $x^{p^n} = x$, we conclude that $f = g$. Conversely, if $\overline{f}\colon \mathbb{F}_{p^n} \to A/N$ is a ring map, we define $f(x)$ for $x \in \mathbb{F}_{p^n}$ as follows. Write $\overline{f}(x) = z + N$ for some $z \in A$, and define $f(x) = z^{p^n}$, One can readily verify that this is independent of the choice of z, and is therefore a ring map that lifts $\overline{f}$.

The proof that R'_k is étale is very similar. If $f, g\colon R'_k \to A$ become equal in A/N, then again $f(x^{p^n}) = g(x^{p^n})$ for all $x \in R'_k$. In particular, since $t_i^{p^n} = t_i$, we see that $f(t_i) = g(t_i)$ for all i, and so $f = g$. Also, given $\overline{f}\colon R'_k \to A/N$, choose $y_i \in A$ such that $\overline{f}(t_i) = y_i + N$. Then define $f\colon R'_k \to A$ by $f(t_i) = y_i^{p^n}$. Since $\overline{f}$ is a ring map, $y_i^{p^n} - y_i \in N$. Hence $(y_i^{p^n})^{p^n} = y_i^{p^n}$. Similarly, since $y_0^{p^n-1} - 1 \in N$, we see that $y_0^{p^{n+1}-p} = 1$, and so y_0 is a unit. Hence f is a ring map lifting $\overline{f}$. □

By combining Theorem 2.7 with Theorem 3.6, Lemma 3.3, and Theorem 3.13, we get the following theorem.

Theorem 3.14. *The map $\Phi\colon E^\vee_* E \to C(\Gamma, E_*)$ is an isomorphism if and only if $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism, where $\Psi\colon K_0E \to C(\Gamma, \mathbb{F}_{p^n})$ is induced by Φ.*

4. Geometric points

In this section, we prove that

$$\Phi\colon E^\vee_* E \to C(\Gamma, E_*)$$

is an isomorphism by calculating the effect of

$$\Psi\colon K_0E \to C(\Gamma, \mathbb{F}_{p^n})$$

on geometric points; that is, on $\mathbb{F}_{p^n}$-algebra homomorphisms to $\overline{\mathbb{F}_p}$.

Proposition 4.1. $\mathbf{Alg}_{\mathbb{F}_{p^n}}(K_0E, \overline{\mathbb{F}_p})$ *is isomorphic to the set of pairs (α, β), where α is an automorphism of the Honda formal group law F_n defined over $\mathbb{F}_{p^n}$, and β is an element of the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$.*

Proof. By Proposition 3.12 and Corollary 3.11, an $\mathbb{F}_{p^n}$-algebra homomorphism from K_0E to $\overline{\mathbb{F}_p}$ is equivalent to an automorphism of F_n defined over $\overline{\mathbb{F}_p}$ together with a ring homomorphism $\mathbb{F}_{p^n} \to \overline{\mathbb{F}_p}$. A ring homomorphism $\mathbb{F}_{p^n} \to \overline{\mathbb{F}_p}$ is equivalent to an element of the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. Also, any endomorphism of F_n over $\overline{\mathbb{F}_p}$ is in fact defined over $\mathbb{F}_{p^n}$ by [Rav86, Theorem A2.2.17]. □

Proposition 4.2. $\mathbf{Alg}_{\mathbb{F}_{p^n}}(C(\Gamma, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p})$ *is isomorphic to* Γ. *The isomorphism takes* $x \in \Gamma$ *to the map that evaluates at* x.

Proof. In view of Lemma 2.2, we have

$$\mathbf{Alg}_{\mathbb{F}_{p^n}}(C(\Gamma, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p}) \cong \lim_U \mathbf{Alg}_{\mathbb{F}_{p^n}}(F(\Gamma/U, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p}),$$

as U runs over the open normal subgroups of Γ. But $F(\Gamma/U, \mathbb{F}_{p^n})$ is generated as an $\mathbb{F}_{p^n}$-algebra by the orthogonal idempotents x^* for $x \in \Gamma/U$, where x^* is the function that takes x to 1 and everything else to 0. Note that $\sum_{x \in \Gamma/U} x^* = 1$. There are no idempotents in $\overline{\mathbb{F}_p}$ except 0 and 1. It follows easily that

$$\mathbf{Alg}_{\mathbb{F}_{p^n}}(F(\Gamma/U, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p}) \cong \Gamma/U$$

where the isomorphism takes $x \in \Gamma/U$ to the map that evaluates at x. Taking inverse limits gives the desired result. □

Thus to proceed any further, we must identify the group Γ. Strickland does this in [Str00, Proposition 4], but we will fill in many missing details.

We first prove some preliminary results about Landweber exact spectra. Recall that, if (A, Γ) is a Hopf algebroid and M and N are Γ-comodules, then $M \wedge N$ denotes the comodule which is isomorphic to the tensor product of the **left** A-modules M and N, with the diagonal coaction of Γ.

Lemma 4.3. *Suppose* F *and* G *are evenly graded Landweber exact spectra. Then the natural map* $MU_*F \wedge MU_*G \to MU_*(F \wedge G)$ *of* MU_*MU*-comodules is an isomorphism.*

Proof. This is actually true for any evenly generated spectra F and G, in the sense of [HS99, Definition 2.10]. Indeed, it is clear for even finite spectra, and then one takes a suitable colimit. Any evenly graded Landweber exact spectrum is evenly generated, by [HS99, Proposition 2.12]. □

Lemma 4.4. *Suppose* F *and* G *are Landweber exact spectra. Then* $F \wedge G$ *is also Landweber exact.*

Proof. Landweber exactness implies that

$$(F \wedge G)_* \cong F_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} G_*$$

as MU_*-modules. Now let (A, Γ) be any Hopf algebroid. An A-module B is called Landweber exact if $B \otimes_A (-)$ takes exact sequences of Γ-comodules to exact sequences of A-modules. This is equivalent to $B \otimes_A \Gamma$ being flat as a right A-module, by [HS03, Lemma 2.2]. Now, if B and C are Landweber exact, then $B \otimes_A \Gamma \otimes_A C$ is obviously Landweber exact since

$$B \otimes_A \Gamma \otimes_A C \otimes_A \Gamma \cong (B \otimes_A \Gamma) \otimes_A (C \otimes_A \Gamma)$$

is flat, as the tensor product of two flat modules. □

Proposition 4.5. *Suppose* F *and* G *are Landweber exact, evenly graded, commutative* MU*-algebra spectra. Let* $\phi_F \colon MU_*F \to F_*$ *denote the map induced by the action of* MU *on* F*, and let* $\eta_F \colon F_* \to MU_*F$ *denote the map induced by the unit of* MU*.*

(1) *The map* $[F, G] \to \mathrm{Hom}_{MU_*}(MU_*F, G_*)$ *that takes* f *to* $\phi_G \circ MU_*f$ *is an isomorphism.*

(2) *The map* $f\colon F \to G$ *is an isomorphism if and only if* $\phi_G \circ MU_* f \circ \eta_F$ *is an isomorphism.*
(3) *The map* $f\colon F \to G$ *is a map of ring spectra if and only if* $\phi_G \circ MU_* f$ *is a map of* MU_**-algebras.*

Proof. Part 1 is proved in [HS99, Corollary 2.17]. For part 2, we note that

$$\phi_G \circ MU_* f \circ \eta_F = \phi_G \circ \eta_G \circ f_* = f_*.$$

For part 3, if f is a map of ring spectra, then $MU_* f$ is a map of MU_*-algebras, so $\phi_G \circ MU_* f$ is also a map of MU_*-algebras. Conversely, suppose $\phi_G \circ MU_* f$ is a map of MU_*-algebras. Then it must take 1 to 1, from which it follows easily that f is compatible with the unit. To see that f is compatible with the multiplication, we first prove that $MU_* f$ is a map of MU_*-algebras. Indeed, consider the diagram below.

$$\begin{tikzcd}
MU_*F \otimes_{MU_*} MU_*F \arrow[r, "MU_*f \otimes MU_*f"] \arrow[d, "\mu"'] & MU_*G \otimes_{MU_*} MU_*G \arrow[r, "\phi_G \otimes \phi_G"] \arrow[d, "\mu"'] & G_* \otimes_{MU_*} G_* \arrow[d, "\mu"] \\
MU_*F \arrow[r, "MU_*f"'] & MU_*G \arrow[r, "\phi_G"'] & G_*
\end{tikzcd}$$

The right-hand square of this diagram is commutative. We want to show that the left-hand square is commutative. Because $MU_*G \cong MU_*MU \otimes_{MU_*} G_*$ is an extended MU_*MU-comodule, it suffices to check that the two composites become equal upon applying ϕ_G, This is equivalent to checking whether the outer boundary of the diagram is commutative. But this is true since $\phi \circ MU_* f$ is a map of MU_*-algebras.

We have now shown that $MU_* f$ is a map of MU_*-algebras. Using the isomorphism of Lemma 4.3, we see that the diagram below is commutative.

$$\begin{tikzcd}
MU_*(F \wedge F) \arrow[r, "MU_*(f \wedge f)"] \arrow[d, "MU_*\mu"'] & MU_*(G \wedge G) \arrow[d, "MU_*\mu"] \\
MU_*F \arrow[r, "MU_*f"'] & MU_*G
\end{tikzcd}$$

But $F \wedge F$ and $G \wedge G$ are also Landweber exact and evenly graded by Lemma 4.4. Hence we can apply part 1 in the form

$$[F \wedge F, G] \cong \mathrm{Hom}_{MU_*MU}(MU_*(F \wedge F), MU_*G)$$

to conclude that $F \to G$ is a map of ring spectra. □

Corollary 4.6. *Suppose F is a Landweber exact, evenly graded, commutative MU-algebra spectra. Then the set of ring spectrum automorphisms of F is isomorphic to the set of F_*-algebra homomorphisms $f\colon F_*F \to F_*$ such that $f\eta_R\colon F_* \to F_*$ is an isomorphism.*

Proof. By Proposition 4.5, the set of ring spectrum automorphisms of F is isomorphic to the set of maps of MU_*-algebras $h\colon MU_*F \to F_*$ such that $h \circ \eta_E$ is an isomorphism. This is equivalent to the set of F_*-algebra homomorphisms

$$f\colon F_*F \cong F_* \otimes_{MU_*} MU_*F \to F_*$$

such that $f\eta_R$ is an isomorphism. □

We can now apply this to our spectrum E. Recall that there is a p-typical formal group law F over E_0, whose classifying map takes v_i to u_i for $i < n$, takes v_n to 1, and takes all other v_i to 0.

Proposition 4.7. *The group Γ is isomorphic to the set of all pairs (τ, ϕ) where τ is an automorphism of the ring E_0 and ϕ is an isomorphism from the formal group F to $\tau_* F$.*

Proof. In view of Corollary 4.6 and the fact that E_* has a unit u in degree 2, we see that Γ is isomorphic to the set of all E_0-algebra maps

$$f\colon E_0E \to E_0$$

such that $f\eta_R$ is an isomorphism. Proposition 3.8 tells us that

$$E_0E \cong E_0 \otimes_V VT[t_0, t_0^{-1}] \otimes_V E_0,$$

Since the tensor product is the coproduct in the category of commutative rings, and $VT[t_0, t_0^{-1}]$ is the representing ring for isomorphisms of formal group laws, we get the desired result. □

Recall that the correspondence between elements of Γ and E_0-algebra maps $f\colon E_0E \to E_0$ for which $f\eta_R$ is an isomorphism takes γ to $(\pi_0\mu) \circ E_0(\gamma)$. The following corollary is then immediate.

Corollary 4.8. *If $\gamma \in \Gamma$ corresponds to (τ, ϕ) as in Proposition 4.7, then the induced map*

$$(\pi_0\mu) \circ E_0(\gamma)\colon\ E_0E \cong E_0 \otimes_V VT[t_0, t_0^{-1}] \otimes_V E_0 \to E_0$$

takes $a \otimes b \otimes c$ to $as_\phi(b)\tau(c)$, where s_ϕ denotes the classifying map of ϕ.

We can now identify Γ, following [Str00, Proposition 4].

Theorem 4.9. *Let Γ_0 denote the automorphism group of the Honda formal group law F_n as a formal group law over $\mathbb{F}_{p^n}$, and let C denote the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. Then $\Gamma \cong \Gamma_0 \rtimes C$. In particular, Γ is isomorphic to the set of pairs (α, β), where α is an automorphism of F_n defined over $\mathbb{F}_{p^n}$, and β is an element of the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$.*

Proof. Proposition 4.7 and Lubin-Tate deformation theory [LT66] show that Γ is isomorphic to the set of all pairs (α, β) where β is an automorphism of $\mathbb{F}_{p^n} \cong E_0/\mathfrak{m}$ and α is an isomorphism of the reduction F_n of F modulo $\mathfrak{m}$ with $\beta_* F_n$. But F_n is defined over $\mathbb{F}_p$, so $\beta_* F_n = F_n$. Hence α is an automorphism of F_n defined over $\mathbb{F}_{p^n}$, and β is an element of C.

There is an obvious action of C on Γ_0; if $\sigma \in C$, then σ induces an isomorphism $\mathbb{F}_{p^n}[[x]] \to \mathbb{F}_{p^n}[[x]]$ that takes $\sum_i c_i x^i$ to $\sum_i c_i^\sigma x^i$. This isomorphism preserves Γ_0. We leave to the reader the check that the multiplication of Γ corresponds to the multiplication on $\Gamma_0 \rtimes C$. □

Combining Theorem 4.9 with Proposition 4.1, we see that the two sets

$$\mathbf{Alg}_{\mathbb{F}_{p^n}}(K_0E, \overline{\mathbb{F}_p}) \text{ and } \mathbf{Alg}_{\mathbb{F}_{p^n}}(C(\Gamma, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p})$$

are abstractly isomorphic. To see that the map $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism, though, we need to understand the map Ψ better. The main point is the following corollary.

Corollary 4.10. *If $\gamma \in \Gamma$ corresponds to (α, β) as in Theorem 4.9, then the map*

$$\Psi\colon \mathbb{F}_{p^n} \otimes_V VT[t_0, t_0^{-1}] \otimes_V \mathbb{F}_{p^n} \cong K_0E \to C(\Gamma, \mathbb{F}_{p^n})$$

has $\Psi(a \otimes b \otimes c)(\gamma) = as_\alpha(b)\beta(c)$, where s_α is the classifying map of α.

Proof. Recall that Ψ is induced by Φ by dividing by $\mathfrak{m}$ and taking the induced map in degree 0. Using the fact that $K_*E = E_*^\vee E/\mathfrak{m}$ (Proposition 8.4 of [HS99]), we see that $\Psi(a)(\gamma)$ is the composite

$$S^0 \xrightarrow{a} K \wedge E \xrightarrow{1\wedge\gamma} K \wedge E \xrightarrow{\mu} K.$$

Since the Lubin-Tate correspondence proceeds by reducing a pair (τ, ϕ) as in Proposition 4.7 modulo $\mathfrak{m}$ to obtain (α, β), this corollary follows from Corollary 4.8. □

We can now complete the proof that Φ is an isomorphism.

Theorem 4.11. *The map*

$$\Phi\colon E_*^\vee E \to C(\Gamma, E_*)$$

of Section 1 is an isomorphism.

Proof. In view of Theorem 3.14, it suffices to show that $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism. By Proposition 4.2, a typical element of $\mathbf{Alg}_{\mathbb{F}_{p^n}}(C(\Gamma, \mathbb{F}_{p^n}), \overline{\mathbb{F}_p})$ is the evaluation at γ map Ev_γ for $\gamma \in \Gamma$. If γ corresponds to the pair (α, β) as in Theorem 4.9, then Corollary 4.10 tells us that $\mathrm{Ev}_\gamma \circ \Psi$ is the $\mathbb{F}_{p^n}$-algebra map

$$\mathrm{Ev}_\gamma \circ \Psi\colon \mathbb{F}_{p^n} \otimes_V VT[t_0, t_0^{-1}] \otimes_V \mathbb{F}_{p^n} \to \overline{\mathbb{F}_p}$$

that takes $a \otimes b \otimes c$ to $as_\alpha(b)\beta(c)$, where s_α is the classifying map of α. Under the isomorphism of Proposition 4.1, this map corresponds to (α, β). Thus $\mathbf{Alg}_{\mathbb{F}_{p^n}}(\Psi, \overline{\mathbb{F}_p})$ is an isomorphism. □

5. Cohomology operations

In this section, we prove that E^*E is isomorphic to the twisted completed group ring $E_*[[\Gamma]]$. The strategy is to construct the commutative square below,

$$\begin{array}{ccc} E_*[[\Gamma]] & \longrightarrow & E^*E \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{E_*}(C(\Gamma, E_*), E_*) & \longrightarrow & \mathrm{Hom}_{E_*}(E_*^\vee E, E_*) \end{array}$$

and to show that the vertical maps are isomorphisms.

We begin with the right-hand vertical map in Section 5.1, and we discuss the left-hand vertical map in Section 5.2. We then complete the proof in Section 5.3.

5.1. Duality for E^*E.

The object of this section is to prove the following theorem.

Theorem 5.1. *Suppose X is a spectrum such that K_*X is evenly graded. Then the natural map*

$$E^*X \xrightarrow{\rho} \mathrm{Hom}_{E_*}(E_*^\vee X, E_*)$$

is an isomorphism.

The natural map in question takes an element $x \in E^m X$ to the homomorphism that takes $y \in E_k^\vee X$ to the composite

$$S^{k-m} \xrightarrow{\Sigma^{-m} y} E \widetilde{\wedge} \Sigma^{-m} X \xrightarrow{1 \widetilde{\wedge} x} E \widetilde{\wedge} E \xrightarrow{\mu} E.$$

One can define an analogous map

$$\rho \colon F^* X \to \operatorname{Hom}_{E_*}(E_*^\vee X, F_*)$$

for any E-module spectrum F.

Note that Theorem 5.1 would be automatic if we had a suitable universal coefficient spectral sequence for $E_*^\vee(-)$. Indeed, since $K_* X$ is evenly graded, $E_*^\vee X$ is pro-free and so projective in the category of L-complete E_*-modules [HS99, Theorem A.9]. We do have a universal coefficient spectral sequence for $E_*(-)$, following Adams' approach [Ada74, Section III.13]. But our hypotheses do not guarantee that $E_* X$ is projective over E_*.

We will prove Theorem 5.1 by working with $(E/J)^*(X)$, where

$$J = (p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$$

is a regular ideal in E_*, and E/J is obtained by taking successive cofibers as in [EKMM97, Chapter V]. The ring E_*/J is a local ring with nilpotent maximal ideal $\mathfrak{m}$. We have the following lemma, which is surely well-known, about modules over such rings.

Lemma 5.2. *Suppose $f \colon M \to N$ is a map of R-modules, where R is a local ring with nilpotent maximal ideal $\mathfrak{m}$.*

1. *f is surjective if and only if $R/\mathfrak{m} \otimes_R f$ is surjective.*
2. *If M and N are flat and $R/\mathfrak{m} \otimes_R f$ is an isomorphism, then f is an isomorphism.*

It follows easily from this lemma that flat R-modules are in fact free, though we do not need this fact.

Proof. Suppose that $R/\mathfrak{m} \otimes_R f$ is surjective. Given y in N, we show by induction on k that there is an x_k in M such that $f(x_k) - y \in \mathfrak{m}^k N$. Since $\mathfrak{m}$ is nilpotent, taking k large enough shows that f is surjective. The case $k = 1$ is clear since $R/\mathfrak{m} \otimes_R f$ is surjective. Now suppose that

$$f(x_k) - y = r_1 z_1 + \cdots + r_j z_j$$

where $r_i \in \mathfrak{m}^k$. For each i, we can find an $w_i \in M$ such that $z_i = f(w_i) + t_i$, where $t_i \in \mathfrak{m}N$, by the base case of the induction. It follows that

$$f(x_k - r_1 w_1 - \cdots - r_j w_j) - y \in \mathfrak{m}^{k+1} N,$$

proving the induction step.

Now suppose M and N are flat. Let K denote the kernel of f. Since N is flat, the sequence

$$0 \to K \to M \xrightarrow{f} N \to 0$$

pure, and so K is flat [Lam99, Corollary 4.86] and $K/\mathfrak{m}K = 0$. Since $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a free module over the field $R/\mathfrak{m}$, we see that $K \otimes_R \mathfrak{m}^k/\mathfrak{m}^{k+1} = 0$ for all k. Tensoring the short exact sequence

$$0 \to \mathfrak{m}^k/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^k \to 0$$

with K, we conclude by induction that $K/\mathfrak{m}^k K = 0$ for all k. Since $\mathfrak{m}$ is nilpotent, we see that $K = 0$. □

With this lemma in hand, we can now analyse $(E/J)^*(X)$.

Proposition 5.3. *Suppose X is a spectrum such that K_*X is evenly graded, and $J = (p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$ is a regular ideal in E_*. Then the natural map*

$$\rho\colon\ (E/J)^*(X) \to \operatorname{Hom}_{E_*}(E_*^\vee X, E_*/J)$$

is an isomorphism.

Proof. Note that

$$\operatorname{Hom}_{E_*}(E_*^\vee X, E_*/J) \cong \operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, E_*/J).$$

and E_*/J is a commutative ring with nilpotent maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$. The plan is thus to use Lemma 5.2.

Since K_*X is evenly graded, so is its dual K^*X. Thus E^*X is pro-free, using the method of [HS99, Proposition 2.5]. In particular, the sequence $(p, u_1, \dots, u_{n-1})$ is a regular sequence on E^*X. It follows from [Mat89, Theorem 16.1] that

$$(p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$$

is also a regular sequence on E^*X, and therefore that $(E/J)^*X \cong (E^*X)/J$, Since E^*X is the completion of a free E_*-module, we conclude that $(E/J)^*X$ is a free E_*/J-module, and that $((E/J)^*X)/\mathfrak{m} \cong K^*X$.

Similarly, because K_*X is evenly graded, $E_*^\vee X$ is evenly graded and pro-free, by the method of [HS99, Proposition 2.5]. The same argument as in the preceding paragraph then shows that $(E/J)_*X \cong (E_*^\vee X)/J$ and is a free E_*/J-module. It follows that $\operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, E_*/J)$ is a product of free modules, and so is flat (since E_*/J is Noetherian). It also follows that $((E/J)_*X)/\mathfrak{m} \cong K_*X$.

Furthermore, there is a natural map

$$\operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, E_*/J) \to \operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, K_*) \cong \operatorname{Hom}_{K_*}(K_*X, K_*).$$

This map is surjective, since $(E_*^\vee X)/J$ is free. Hence we get an induced surjection

$$(\operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, E_*/J))/\mathfrak{m} \to \operatorname{Hom}_{K_*}(K_*X, K_*).$$

We claim this map is an isomorphism. To see this suppose $f\colon\ (E_*^\vee X)/J \to E_*/J$ is a homomorphism that goes to zero as a map to K_*. This means that $f(x) \in \mathfrak{m}$ for all x. In particular, if we let $\{e_i\}$ be a generating set for the free module $(E_*^\vee X)/J$, we can write

$$f(e_i) = px_{i0} + u_1 x_{i1} + \cdots u_{n-1} x_{i,n-1}$$

for some elements x_{ij} in E_*/J. But then we can define homomorphisms

$$g_j\colon\ (E_*^\vee X)/J \to E_*/J$$

for $j = 0, 1, \dots, n-1$ by $g_j(e_i) = x_{ij}$. This gives

$$f = pg_0 + u_1 g_1 + \cdots u_{n-1} g_{n-1},$$

and so $f \in \mathfrak{m} \operatorname{Hom}_{E_*/J}((E_*^\vee X)/J, E_*/J)$, as required.

It now follows from an easy diagram chase that if we tensor the map

$$(E/J)^*X \to \operatorname{Hom}_{E_*}(E_*^\vee X, E_*/J)$$

with $E_*/\mathfrak{m}$, we get the isomorphism

$$K^*X \cong \operatorname{Hom}_{K_*}(K_*X, K_*).$$

Lemma 5.2 completes the proof. □

We now prove Theorem 5.1.

Proof of Theorem 5.1. Choose a sequence of ideals

$$\cdots \subseteq J_k \subseteq J_{k-1} \subseteq \cdots \subseteq J_1$$

such that each J_k is of the form $(p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$ and the J_k converge to 0 in the $\mathfrak{m}$-adic topology. We have a commutative square

$$\begin{CD} E^*X @>{\rho}>> \operatorname{Hom}_{E_*}(E_*^{\vee}X, E_*) \\ @VVV @VVV \\ \lim_k (E/J_k)^*(X) @>>{\lim \rho_k}> \lim_k \operatorname{Hom}_{E_*}(E_*^{\vee}X, E_*/J_k). \end{CD}$$

The right-hand vertical map is an isomorphism because E_* is complete. To understand the left-hand vertical map, we apply the Milnor exact sequence

$$0 \to \lim^1 (E/J_k)^{*+1}X \to E^*X \to \lim (E/J_k)^*X \to 0.$$

Since K_*X is evenly graded, $(E/J_k)^*X \cong (E_*^{\vee}X)/J_k$ as we have seen in the proof of Proposition 5.3. Thus the $\lim^1$-terms vanish, and so the left-hand vertical map is an isomorphism. Proposition 5.3 implies that the bottom horizontal map is an isomorphism as well, completing the proof. □

5.2. **Duality for** $C(G,R)$**.** We now turn our attention to the twisted completed group ring $E_*[[\Gamma]]$.

In general, whenever G is a profinite group and $\mathfrak{a}$ is an ideal in a ring R, we can define the R-module

$$R[[G]] = \lim_k \lim_U (R/\mathfrak{a}^k)[G/U].$$

We then have the following theorem.

Theorem 5.4. *Let G be a profinite group and $\mathfrak{a}$ an ideal in a commutative ring R such that R is complete in the $\mathfrak{a}$-adic topology. Then the natural map*

$$R[[G]] \to \operatorname{Hom}_R(C(G,R), R)$$

is an isomorphism of R-modules.

Proof. There is a natural map

$$\alpha \colon R[G] \to \operatorname{Hom}_R(C(G,R), R),$$

where $\alpha(r[g])(f) = rf(g)$. It is clear that α is an isomorphism when G is finite. Hence we get an induced natural isomorphism

$$R[[G]] = \lim_{k,U} (R/\mathfrak{a}^k)[G/U] \to \lim_{k,U} \operatorname{Hom}_{R/\mathfrak{a}^k}(C(G/U, R/\mathfrak{a}^k), R/\mathfrak{a}^k).$$

We must show that the right-hand side of this isomorphism is naturally isomorphic to $\operatorname{Hom}_R(C(G,R), R)$. To see this, we first use Proposition 2.3 and Proposition 2.4 to conclude that $C(G,R)/\mathfrak{a}^k \cong C(G, R/\mathfrak{a}^k)$ This fact together with Lemma 2.2 gives us the following chain of isomorphisms

$$\begin{aligned} \operatorname{Hom}_R(C(G,R), R) &\cong \lim_k \operatorname{Hom}_R(C(G,R), R/\mathfrak{a}^k) \\ &\cong \lim_k \operatorname{Hom}_{R/\mathfrak{a}^k}(C(G, R/\mathfrak{a}^k), R/\mathfrak{a}^k) \\ &\cong \lim_k \lim_U \operatorname{Hom}_{R/\mathfrak{a}^k}(C(G/U, R/\mathfrak{a}^k), R/\mathfrak{a}^k), \end{aligned}$$

completing the proof. □

Now, when G acts continuously on R, $\mathrm{Hom}_R(C(G,R),R)$ is the dual of the Hopf algebroid $C(G,R)$, so is an R-algebra. In view of Theorem 5.4, then, $R[[G]]$ should also be an algebra, in analogy with the group ring, but in a way that takes into account the action of G on R.

To see how this works, assume first that an arbitrary group G acts on a commutative ring R by ring isomorphisms. Then the **twisted group ring** $R[G]$ is the free R-module generated by the elements of G, with multiplication defined by $(a[g])(b[h]) = ab^g[gh]$. We would like to realize $R[[G]]$ as a completion of the twisted group ring $R[G]$ with respect to a suitable family of ideals. For this, we need to assume G is a profinite group, R is a local ring with maximal ideal $\mathfrak{m}$ that is complete in the $\mathfrak{m}$-adic topology, and G acts continuously on R, and, even better, acts through a finite quotient on $R/\mathfrak{m}^k$ for all k. That is, we need to assume that, for all k, there is an open normal subgroup U_k of G such that U_k acts trivially on $R/\mathfrak{m}^k$. Note that this is automatic from continuity if $R/\mathfrak{m}^k$ is finite, or, in the graded case, finite in each degree, as is true for $R = E_*$. Indeed, in this case $\mathrm{Aut}(R/\mathfrak{m}^k)$ is finite, so the homomorphism $G \to \mathrm{Aut}(R/\mathfrak{m}^k)$ must factor through a finite quotient.

Assuming that G does act through a finite quotient on each $R/\mathfrak{m}^k$, we define ideals $I(k,U)$ of $R[G]$ for each integer k and each open normal subgroup U of G such that U acts trivially on $R/\mathfrak{m}^k$. The ideal $I(k,U)$ is the kernel of the surjection $R[G] \to (R/\mathfrak{m}^k)[G/U]$ to the twisted group ring. In more concrete terms, $I(k,U)$ is the set of all elements $\sum a_g[g]$ in $R[G]$ such that for every coset C of U in G, we have $\sum_{g\in C} a_g \in \mathfrak{m}^k$. We then the **twisted completed group ring** $R[[G]]$ to be the completion of $R[G]$ with respect to the ideals $I(k,U)$, so that

$$R[[G]] = \lim_{k,U}(R/\mathfrak{a}^k)[G/U].$$

This is the same R-module as we defined above, since we can take this inverse limit over arbitrary pairs (k,U). Note that the natural map $R[G] \to R[[G]]$ is injective since $\bigcap_k \mathfrak{m}^k = 0$. All the statements above work when R is a graded ring, as long as $\mathfrak{m}$ is homogeneous and the action of G preserves the grading.

With these definitions, the isomorphism of Theorem 5.4 is in fact an isomorphism of R-algebras, using the mulitplication on $\mathrm{Hom}_R(C(G,R),R)$ dual to the Hopf algebroid structure on $C(G,R)$. We do not need this result, so we leave the proof to the interested reader.

5.3. **The isomorphism.** We can now compute E^*E.

Theorem 5.5. *The inclusion $\Gamma \to E^0E$ induces an E_*-algebra isomorphism*

$$E_*[[\Gamma]] \to E^*E.$$

Proof. There is a map $E_* \to E^*E$ that takes $r \in E_k$ to $\widetilde{r} \in E^kE$, where $\widetilde{r}$ is the composite

$$E = S^0 \wedge E \xrightarrow{r\wedge 1} \Sigma^k E \wedge E \xrightarrow{\Sigma^k \mu} \Sigma^k E.$$

The inclusion of Γ then certainly induces a map $\alpha\colon E_*[\Gamma] \to E^*E$, defined by letting $\alpha(r[\gamma])$ be the composite $\widetilde{r}\circ\gamma$. To see that α is an E_*-algebra homomorphism, use the commutative diagram below.

$$\begin{tikzcd}
E \arrow[r, "\cong"] \arrow[d, "\gamma"'] & S^0 \wedge E \arrow[r, "r\wedge 1"] \arrow[d, "1\wedge\gamma"'] & \Sigma^k E \wedge E \arrow[r, "\Sigma^k \mu"] \arrow[d, "\Sigma^k\gamma\wedge\gamma"'] & \Sigma^k E \arrow[d, "\Sigma^k\gamma"] \\
E \arrow[r, "\cong"'] & S^0 \wedge E \arrow[r, "r^\gamma\wedge 1"'] & \Sigma^k E \wedge E \arrow[r, "\Sigma^k\mu"'] & \Sigma^k E
\end{tikzcd}$$

This diagram commutes because γ is a map of ring spectra, and it shows that $\alpha([\gamma]r) = (\alpha[\gamma])(\alpha r)$, so that α is an E_*-algebra homomorphism.

Now, E^*E is complete with respect to the $\mathfrak{m}$-adic topology, so to show that α extends to an E_*-algebra homomorphism

$$\beta\colon\ E_*[[\Gamma]] \to E^*E,$$

it suffices to show that α is continuous. That is, given k, we must find an m and U such that $\alpha I(m,U) \subseteq \mathfrak{m}^k E^*E$. But the action of Γ on E^*E is continuous, so there is a U such that U preserves $\mathfrak{m}^k E^*E$. This means that the composite

$$E_*[\Gamma] \xrightarrow{\alpha} E^*E \to (E^*E)/\mathfrak{m}^k$$

factors through $(E_*/\mathfrak{m}^k)[\Gamma/U]$, and so $\alpha I(k,U) \subseteq \mathfrak{m}^k E^*E$, as required.

Now consider the diagram below.

$$\begin{tikzcd}
E_*[[\Gamma]] \arrow[r, "\beta"] \arrow[d] & E^*E \arrow[d, "\rho"] \\
\mathrm{Hom}_{E_*}(C(\Gamma, E_*), E_*) \arrow[r] & \mathrm{Hom}_{E_*}(E^\vee_* E, E_*)
\end{tikzcd}$$

The left-hand vertical map is the isomorphism of Theorem 5.4, the right-hand vertical map is the isomorphism of Theorem 5.1, and the bottom horizontal map is dual to the isomorphism of Theorem 4.11. One can easily check that the diagram is commutative. Indeed, it suffices to check that the element $[\gamma] \in E_*[\Gamma]$ goes to the same place under both composites. In fact, it goes to the map that takes $x \in E^\vee_k E$ to the composite

$$S^0 \xrightarrow{x} \Sigma^k E \widetilde{\wedge} E \xrightarrow{\Sigma^k(1\widetilde{\wedge}\gamma)} \Sigma^k E \widetilde{\wedge} E \xrightarrow{\Sigma^k\mu} \Sigma^k E.$$

Hence β is an isomorphism. □

6. Hopf algebroid structure

The object of this section is to show that $(E_*, E^\vee_* E)$ and $(E_*, C(\Gamma, E_*))$ are both graded formal Hopf algebroids, and that the isomorphism Φ of Theorem 4.11 preserves the Hopf algebroid structure. We begin by defining precisely what we mean by a graded formal Hopf algebroid. We then show that $(E_*, E^\vee_* E)$ is a graded formal Hopf algebroid in Section 6.2, then discuss $C(\Gamma, E_*)$ in Section 6.3. We show that Φ preserves the Hopf algebroid structure in Section 6.4.

6.1. **Graded formal Hopf algebroids.** A graded formal Hopf algebroid should be a cogroupoid object in the category of graded formal rings. Recall from [Str99, Section 4] that a **formal ring** A is a topological commutative ring A such that cosets of open ideals form a basis for the topology on A, and such that $A \cong \lim A/I$ as I runs through the open ideals. Morphisms in the category of formal rings are continuous homomorphisms. The category of formal rings has all finite colimits.

The pushout $S\widehat{\otimes}_R T$ of two continuous homomorphisms $R \to S$ and $R \to T$ is the completion of the tensor product $S \otimes_R T$ with respect to the topology defined by the ideals $I \otimes_R T + S \otimes_R J$, where I and J run through open ideals in S and T. Note that this definition of $\widehat{\otimes}$ does not conflict with our previous use of the notation, in Proposition 2.4. We can thus define a **formal Hopf algebroid** to be a cogroupoid object in the category of formal rings. The major difference between a formal Hopf algebroid (A, Γ) and a Hopf algebroid is that the diagonal has the form $\Delta\colon \Gamma \to \Gamma\widehat{\otimes}\Gamma$, and so all diagrams involving Δ must be similarly changed.

Some subtleties arise in the graded case. We define a **graded formal ring** to be a graded ring A equipped with a family $\{I_j\}$ of homogeneous ideals such that for all j, j' there is a j'' such that $I_{j''} \subseteq I_j \cap I_{j'}$ and such that A is the inverse limit, in the category $\mathbf{Rings}_{gr}$ of graded rings, of the A/I_j. This means that $A_k \cong \lim_j (A/I_j)_k$ for all k. The family of ideals $\{I_j\}$ will sometimes be referred to as the **topology** on A. Maps of graded formal rings $R \to S$ are maps of graded rings such that each $R_k \to S_k$ is continuous (in the inverse limit topology).

Note that the forgetful functor from $\mathbf{Rings}_{gr}$ to $\mathbf{Rings}$ does **NOT** preserve inverse limits. In particular, if k is a field and x has degree 2, the ring $k[x]$ is a graded formal ring in the x-adic topology, but it is not a formal ring in the x-adic topology.

Nevertheless, the category of graded formal rings has all finite colimits, where the pushout of $R \to S$ and $R \to T$ is $S\widehat{\otimes}_R T$, the completion in the graded sense of the graded ring $S \otimes_R T$ with respect to the ideals $I \otimes T + S \otimes J$, where I and J are the homogeneous ideals that define the formal structure on R and S. Hence we can define a **graded formal Hopf algebroid** to be a cogroupoid object in the category of graded formal rings.

From an algebro-geometric perspective, it would be much more natural to consider formal graded rings, rather than graded formal rings. Here a **formal graded ring** would be a formal ring R equipped with a coaction of the Hopf algebra $\mathbb{Z}[u, u^{-1}]$. That is, the "grading" would actually be a continuous ring homomorphism $R \to R\widehat{\otimes}\mathbb{Z}[u, u^{-1}]$ that is coassociative and counital. The drawback of this approach for us is that this is just not how we think of completions in algebraic topology. For example, when we form $(v_n^{-1}BP_*)_{I_n}$, we do not allow the element $\sum_{k=1}^{\infty} v_1^k$. It is possible that it would be better to allow such elements, giving us homotopy groups that are only graded in this weak sense. We will stick with graded formal rings though.

6.2. $E^\vee_* E$ **as a graded formal Hopf algebroid.** We now show that $(E_*, E^\vee_* E)$ is a graded formal Hopf algebroid. The topology is defined by the powers of the maximal ideal $\mathfrak{m}$ of E_*. We have seen that $E^\vee_* E$ is pro-free in Proposition 2.1, and so $E^\vee_* E$ is a graded formal ring. The usual proof that (R_*, R_*R) is a Hopf algebroid [Rav86, Proposition 2.2.8] depends on knowing that the natural map

$$R_*R \otimes_{R_*} R_*R \to R_*(R \wedge R)$$

is an isomorphism. If we are to apply the same argument in this case, we need to know that

$$E^\vee_* E \widehat{\otimes}_{E_*} E^\vee_* E \to E^\vee_*(E \widetilde{\wedge} E)$$

is an isomorphism.

Proposition 6.1. *Suspose X is a spectrum such that K_*X is concentrated in even dimensions. Then the natural map*

$$\sigma\colon\ (E^\vee_* E \otimes_{E_*} E^\vee_* X)^\wedge_{\mathfrak{m}} \to E^\vee_*(E\widetilde{\wedge}X)$$

is an isomorphism.

Proof. Since K is a field spectrum, the natural map

$$\rho\colon\ K_*E \otimes_{K_*} K_*X \to K_*(E \wedge X) = K_*(E\widetilde{\wedge}X)$$

is an isomorphism. In particular, $K_*(E\widetilde{\wedge}X)$ is concentrated in even dimensions. The method of [HS99, Proposition 8.4] then guarantees that $E^\vee_* X$ and $E^\vee_*(E\widetilde{\wedge}X)$ are pro-free, as is $E^\vee_* E$. If $M \cong F^\wedge_{\mathfrak{m}}$ and $N \cong (F')^\wedge_{\mathfrak{m}}$ are pro-free modules, then $(M \otimes N)^\wedge_{\mathfrak{m}}$ is also pro-free, since it is isomorphic to the completion of $F \otimes_{E_*} F'$. Hence $(E^\vee_* E \otimes_{E_*} E^\vee_* X)^\wedge_{\mathfrak{m}}$ is pro-free as well. In view of Lemma 2.6, to prove the proposition it suffices to check that $\sigma/\mathfrak{m}$ is an isomorphism. Using [HS99, Proposition 8.4(e)] and the fact that $\mathfrak{m}$ is an invariant ideal in E_*E, we conclude that $\sigma/\mathfrak{m} \cong \rho$, and so σ is an isomorphism. □

Corollary 6.2. *Suppose X is a ring spectrum such that K_*X is concentrated in even dimensions. Then the natural map*

$$E^\vee_* X \widehat{\otimes}_{E_*} E^\vee_* X \to E^\vee_*(E\widetilde{\wedge}X)$$

is an isomorphism.

Proof. Note that $E^\vee_* X$ is a formal ring with respect to the ideals $\mathfrak{m}^n$, since $E^\vee_* X$ is pro-free [HS99, Proposition 8.4(f)]. This result then follows from Proposition 6.1 because $\mathfrak{m}$ is an invariant ideal in E_*. □

Recall that if R is a flat commutative ring spectrum, then (R_*, R_*R) is a graded Hopf algebroid [Rav86, Proposition 2.2.8]. The only use of the flatness of R in this argument is in the isomorphisms

$$R_*X \otimes_{R_*} R_*Y \to R_*(X \wedge Y)$$

in case $X = Y = R$, needed to define the diagonal, and in case $X = R, Y = R \wedge R$ and $X = R \wedge R$, $Y = R$, needed to prove coassociativity. We can repeat this argument using the isomorphisms of Corollary 6.2 to deduce the following theorem,

Theorem 6.3. $(E_*, E^\vee_* E)$ *is a graded formal Hopf algebroid.*

6.3. $C(\Gamma, E_*)$ **as a graded formal Hopf algebroid.** The object of this section is to show that $(E_*, C(\Gamma, E_*))$ is a graded formal Hopf algebroid. We will work more generally with a profinite group G acting continuously on a graded ring R that is complete in the $\mathfrak{a}$-adic topology, where $\mathfrak{a}$ is a homogeneous ideal.

We begin with a very simple case. Given a group G, the **constant group scheme** G is the functor on commutative rings defined by $G(S) = C(\operatorname{Spec} S, G)$. If S has no nontrivial idempotents, then $\operatorname{Spec} S$ is connected, and so $G(S) = G$, consisting only of the constant maps. The constant group scheme G is in fact the coproduct in the category of schemes of $|G|$-many copies of $\operatorname{Spec} \mathbb{Z}$. In particular, it is not affine when G is infinite, because it is not quasi-compact.

However, when G is finite, we have the following well-known proposition.

Proposition 6.4. *Suppose G is a finite group. Then the constant group scheme G is affine and represented by the Hopf algebra $F(G, \mathbb{Z})$ of functions from G to $\mathbb{Z}$.*

This proposition follows from the fact that $F(G,\mathbb{Z})$ is the product of $|G|$-many copies of $\mathbb{Z}$. The structure maps of the Hopf algebra $F(G,\mathbb{Z})$ are defined as follows. The counit $\epsilon\colon F(G,\mathbb{Z}) \to \mathbb{Z}$ is evaluation at the identity element e of G, the conjugation χ is induced by the inverse map in G, and the diagonal Δ is the composite

$$F(G,\mathbb{Z}) \to F(G\times G,\mathbb{Z}) \xrightarrow{\tau_G^{-1}} F(G,\mathbb{Z})\otimes F(G,\mathbb{Z})$$

where the first map is induced by the multiplication map of G and τ_G is the isomorphism defined by $\tau_G(f\otimes f')(g,h) = f(g)f'(h)$.

We can generalize Proposition 6.4 to profinite groups as well.

Proposition 6.5. *Suppose G is a profinite group. Then the group-valued functor on commutative rings defined by $G(S) = C(\operatorname{Spec} S, G)$, where G is given the profinite topology, is an affine group scheme represented by the Hopf algebra $C(G,\mathbb{Z})$ of continuous functions from G to $\mathbb{Z}$.*

Note that if S has no nontrivial idempotents, then $G(S) = G$, just as for the constant group scheme. There is a natural map from the constant group scheme to this profinite group scheme G, because every map that is continuous to the discrete topology on G is also continuous to the profinite topology, and this map is an isomorphism on every S with no nontrivial idempotents. But it is not an isomorphism on all S unless G is finite, since the constant group scheme is not affine when G is infinite but the profinite group scheme is, according to Proposition 6.5.

Proof. We simply compute

$$\begin{aligned} G(S) = C(\operatorname{Spec} S, G) &\cong \lim\nolimits_U C(\operatorname{Spec} S, G/U) \\ \cong \lim\nolimits_U \mathbf{Rings}(F(G/U,\mathbb{Z}), S) &\cong \mathbf{Rings}(C(G,\mathbb{Z}), S). \end{aligned}$$

This shows that G is affine and represented by $C(G,\mathbb{Z})$. Since G is visibly a group-valued functor, $C(G,\mathbb{Z})$ is a Hopf algebra. □

Note that $C(G,\mathbb{Z})$ is the colimit of the Hopf algebras $F(G/U,Z)$. One can describe the structure maps in a similar fashion. The counit is again evaluation at e, the conjugation is induced by the (continuous) inverse mapping on G, and the diagonal is the composite

$$C(G,\mathbb{Z}) \to C(G\times G,\mathbb{Z}) \xrightarrow{\tau_G^{-1}} C(G,\mathbb{Z})\otimes C(G,\mathbb{Z}),$$

where τ_G is again defined by $\tau_G(f\otimes f')(g,h) = f(g)f'(h)$. The map τ_G is the colimit of the isomorphisms $\tau_{G/U}$, so is an isomorphism.

We now show how to build the Hopf algebroid $(R, C(G,R))$ from the Hopf algebra $C(G,\mathbb{Z})$. Recall that a Hopf algebroid is a cogroupoid object in the category of commutative rings [Rav86, Appendix 1]. In general, a cogroupoid object in a category $\mathcal{C}$ is a pair of objects (A,Γ) such that $(\mathcal{C}(A,X), \mathcal{C}(\Gamma,X))$ is a groupoid that is natural in X for all objects X of $\mathcal{C}$. Here $\mathcal{C}(A,X)$ is the object set of the groupoid and $\mathcal{C}(\Gamma,X)$ is the morphism set. When $\mathcal{C}$ has finite colimits, (A,Γ) is a cogroupoid object if and only if there are structure maps $\eta_L,\eta_R\colon A\to\Gamma$, $\epsilon\colon \Gamma\to A$, $\chi\colon\Gamma\to\Gamma$, and $\Delta\colon \Gamma\to\Gamma\amalg_A\Gamma$ analogous to the structure maps of a Hopf algebroid [Rav86, Appendix 1], where $\Gamma\amalg_A\Gamma$ denotes the pushout of $\Gamma \xleftarrow{\eta_R} A \xrightarrow{\eta_L} \Gamma$. These structure maps are required to satisfy certain axioms analogous to those of [Rav86, Definition A1.1.1].

One of these axioms deserves special attention, because Ravenel's description of it is not optimal. Define $\Gamma \amalg_A \widetilde{\Gamma}$ to be the pushout where A acts on the right factor of Γ though η_R instead of η_L; simmilarly, define $\widetilde{\Gamma} \amalg_A \Gamma$ to be the pushout where A acts on the left factor through η_L instead of η_R. Then axiom (f) of Ravenel's definition [Rav86, A1.1.1], which implies that the composition of a map and its inverse is the appropriate identity, should be rephrased to say that the following diagram, and a similar diagram involving η_L and $1 \amalg \chi$, are commutative.

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, equal] & \Gamma \amalg_A \Gamma \arrow[r, "\chi \amalg 1"] & \widetilde{\Gamma} \amalg_A \Gamma \arrow[d, "\nabla"] \\
\Gamma \arrow[r, "\epsilon"'] & A \arrow[r, "\eta_R"'] & \Gamma
\end{tikzcd}$$

Here ∇ denotes the fold map, which is the identity on each factor of Γ.

Here is a general result on constructing cogroupoid objects from actions of cogroups.

Proposition 6.6. *Suppose $\mathcal{C}$ is a category with finite colimits and C is a cogroup object of $\mathcal{C}$ that coacts on the right on an object A of $\mathcal{C}$. Then $(A, A \amalg G)$ is a cogroupoid object of $\mathcal{C}$.*

A cogroupoid object of this form is sometimes called a **split cogroupoid**. In the case of Hopf algebroids, this recovers the definition of a split Hopf algebroid given in [Rav86, Definition A1.1.22].

Proof. Let 0 denote the initial object of $\mathcal{C}$. Since C is a cogroup, it comes with maps $\Delta\colon C \to C \amalg C$, $\epsilon\colon C \to 0$, and $\chi\colon C \to C$ playing the role of the diagonal, counit, and conjugation. There is also a coassociative and counital map $\psi\colon A \to A \amalg C$ giving the coaction of C. We then define the left unit $\eta_L\colon A \to A \amalg C$ to be the structure map i_1 of the coproduct, the right unit $\eta_R\colon A \to A \amalg C$ to be the coaction ψ, and the counit $\epsilon_A\colon A \amalg C \to A$ to be $1 \amalg \epsilon$. It is then obvious that $\epsilon_A \eta_L = 1_A$, and $\epsilon_A \eta_R = 1_A$ because ψ is counital. We define the conjugation $\chi_A\colon A \amalg C \to A \amalg C$ to be ψ on A and $i_2 \chi$ on C. Then $\chi_A \eta_L = \eta_R$ by definition, and $\chi_A \eta_R = \eta_L$ since $\chi^2 = 1$. Also, $\chi_A^2 = 1_{A \amalg C}$ for the same reasons. Finally, the diagonal Δ_A is the composite

$$A \amalg C \xrightarrow{1 \amalg \Delta} A \amalg C \amalg C \cong (A \amalg C) \amalg_C (A \amalg C),$$

The fact that Δ is coassociative, counital, and compatible with χ implies the same facts for Δ_A. □

We can now apply this to $(R, C(G, R))$,

Theorem 6.7. *Suppose G is a profinite group acting continuously on a ring R that is complete in the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a}$. Then $(R, C(G, R))$ is a split formal Hopf algebroid.*

This theorem is also true in the graded case, where the action of G must preserve the grading, the ideal $\mathfrak{a}$ must be homogeneous, and R need only be complete in the graded sense.

Proof. Think of $C(G, \mathbb{Z})$ as a cogroup object in the category of formal rings, where the topology is trivial. Recall from Proposition 2.4 that there is an isomorphism

$$\sigma\colon R \widehat{\otimes} C(G, \mathbb{Z}) \to C(G, R).$$

Define a coaction ψ of the cogroup $C(G,\mathbb{Z})$ on R as the composite

$$R \xrightarrow{\eta_R} C(G,R) \xrightarrow{\sigma^{-1}} R\widehat{\otimes}C(G,/Z),$$

where $\eta_R(r)(g) = r^g$. It is easy to check that σ is counital. To see that ψ is coassociative, we use the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, "\eta_R"] \arrow[d, "\eta_R"'] & C(G,R) \arrow[d, "\mu^*"'] & R\widehat{\otimes}C(G,\mathbb{Z}) \arrow[l, "\sigma"'] \arrow[d, "1\widehat{\otimes}\mu^*"] \\
C(G,R) \arrow[r, "\beta"] & C(G\times G,R) & R\widehat{\otimes}C(G\times G,\mathbb{Z}) \arrow[l, "\sigma"'] \\
R\widehat{\otimes}C(G,\mathbb{Z}) \arrow[u, "\sigma"] \arrow[r, "\eta_R\widehat{\otimes}1"'] & C(G,R)\widehat{\otimes}C(G,\mathbb{Z}) \arrow[u, "\alpha"] & R\widehat{\otimes}C(G,\mathbb{Z})\widehat{\otimes}C(G,\mathbb{Z}) \arrow[l, "\sigma\widehat{\otimes}1"] \arrow[u, "1\widehat{\otimes}\tau"']
\end{tikzcd}$$

In this diagram, $\mu\colon G\times G \to G$ denotes the multiplication map, $\beta\colon C(G,R) \to C(G\times G,R)$ is defined by $(\beta f)(g,h) = f(h)^g$, and $\alpha\colon C(G,R)\widehat{\otimes}C(G,\mathbb{Z}) \to C(G\times G,R)$ is defined by $\alpha(f\otimes f')(g,h) = f'(h)f(g)$. The reader can check that this diagram is in fact commutative. All the maps that go either left or up are isomorphisms, using Proposition 2.3 and the fact that τ is an isomorphism. Hence we can reverse those arrows, and then the equality of the outer composites shows that η_R is coassociative. Hence Proposition 6.6 completes the proof. $\square$

We now describe the structure maps of $(R, C(G,R))$. The left unit is the inclusion of the constant functions, and the right unit η_R is defined by $\eta_R(r)(g) = r^g$. The counit is evaluation at e, and the conjugation χ is defined by $(\chi f)(g) = f(g^{-1})^g$. We have the commutative diagram below,

$$\begin{tikzcd}
R\widehat{\otimes}C(G,\mathbb{Z}) \arrow[r, "1\widehat{\otimes}\mu^*"] \arrow[d, "\sigma"'] & R\widehat{\otimes}C(G\times G,\mathbb{Z}) \arrow[d, "\sigma"'] & R\widehat{\otimes}C(G,\mathbb{Z})\otimes C(G,\mathbb{Z}) \arrow[l, "1\widehat{\otimes}\tau"'] \arrow[d, "\sigma\widehat{\otimes}\sigma"] \\
C(G,R) \arrow[r, "\mu^*"'] & C(G\times G,R) & C(G,R)\widehat{\otimes}_R C(G,R) \arrow[l, "\tau"]
\end{tikzcd}$$

when the τ on the bottom line is defined to be the completion of the map defined by $\tau(f\otimes f')(g,h) = f(g)f'(h)^g$. Hence τ is an isomorphism, and the diagonal is the composite of the bottom line after reversing τ.

6.4. Φ is a map of graded formal Hopf algebroids.

Theorem 6.8. *The map*

$$(1,\Phi)\colon (E_*, E^\vee_* E) \to (E_*, C(\Gamma, E_*))$$

is a map of graded formal Hopf algebroids.

Proof. we begin by showing that $\Phi\eta_L = \eta_L$. Indeed, if $a\in E_m$ and $\gamma\in\Gamma$, then we have

$$(\Phi\eta_L(a))(\gamma) = \mu\circ(1\widetilde{\wedge}\gamma)\circ(1\widetilde{\wedge}\eta)\circ a = \mu\circ(1\widetilde{\wedge}\eta)\circ a = a = (\eta_L(a)(\gamma),$$

as required. Here we have used the fact that $\gamma\circ\eta = \eta$, since γ is a map of ring spectra, Similarly, we have

$$(\Phi\eta_R(a))(\gamma) = \mu(1\widetilde{\wedge}\gamma)(\eta\widetilde{\wedge}1)a = \mu(\eta\widetilde{\wedge}1)\gamma\circ a = \gamma\circ a = a^\gamma = (\eta_R(a))(\gamma),$$

To see that Φ commutes with ϵ, we have

$$\epsilon\Phi(a) = (\Phi a)(e) = \mu \circ (1\widetilde{\wedge} e) \circ a = \mu \circ a = \epsilon(a).$$

And to see that Φ commutes with χ we compute:

$$\begin{aligned}[\chi\Phi(a)](\gamma) = [\Phi(a)(\gamma^{-1})]^{\gamma} &= \gamma \circ \mu \circ (1\widetilde{\wedge}\gamma^{-1}) \circ a \\ = \mu \circ (\gamma\widetilde{\wedge}\gamma) \circ (1\widetilde{\wedge}\gamma^{-1}) \circ a &= \mu \circ (\gamma\widetilde{\wedge}1) \circ a \\ = \mu \circ (1\widetilde{\wedge}\gamma) \circ T \circ a &= [\Phi(\chi a)](\gamma).\end{aligned}$$

We are left with proving that Φ is compatible with Δ in the sense that $(\Phi \otimes \Phi)^{\wedge}_{\mathfrak{m}} \circ \Delta = \Delta \circ \Phi$. We do this by constructing the commutative diagram below.

$$\begin{tikzcd}
E^{\vee}_{*}E \arrow[r, "\Phi"] \arrow[d, "\rho"'] & C(\Gamma, E_{*}) \arrow[d, "m^{*}"] \\
E^{\vee}_{*}(E\widetilde{\wedge}E) \arrow[r, "\Phi'"] & C(\Gamma\times\Gamma, E_{*}) \\
(E^{\vee}_{*}E \otimes_{E_{*}} E^{\vee}_{*}E)^{\wedge}_{\mathfrak{m}} \arrow[u, "\tau", "\cong"'] \arrow[r, "(\Phi\otimes\Phi)^{\wedge}_{\mathfrak{m}}"'] & (C(\Gamma, E_{*}) \otimes_{E_{*}} C(\Gamma, E_{*}))^{\wedge}_{\mathfrak{m}} \arrow[u, "\cong", "\tau"']
\end{tikzcd} \tag{6.9}$$

Here ρ is induced by $1\widetilde{\wedge}\eta\widetilde{\wedge}1$, and of course the vertical composites (after inverting τ) are the diagonal maps. The map Φ' will be defined analogously to Φ. Indeed, we have a map

$$\sigma' \colon \Gamma \times \Gamma \times E^{\vee}_{*}(E\widetilde{\wedge}E) \to E_{*}$$

that takes the triple (γ, γ', a), where $a \in E^{\vee}_{m}(E\widetilde{\wedge}E)$ to the composite

$$S^{m} \xrightarrow{a} E\widetilde{\wedge}E\widetilde{\wedge}E \xrightarrow{1\widetilde{\wedge}\gamma\widetilde{\wedge}\gamma'} E\widetilde{\wedge}E\widetilde{\wedge}E \xrightarrow{1\widetilde{\wedge}1\widetilde{\wedge}\gamma} E\widetilde{\wedge}E\widetilde{\wedge}E \xrightarrow{1\widetilde{\wedge}\mu} E\widetilde{\wedge}E \xrightarrow{\mu} E.$$

Once again the results of [HS99, Section 11] guarantee that this map is continuous when every set of maps is given its natural topology. Thus the adjoint of σ' is our desired map

$$\Phi' \colon E^{\vee}_{*}(E \wedge E) \to C(\Gamma \times \Gamma, E_{*}).$$

Now we need to check that the two squares in our diagram (6.9) commute. For the top square, we have

$$\begin{aligned}(\Phi'\rho(a))(\gamma,\gamma') &= \mu \circ (1\widetilde{\wedge}\mu) \circ (1\widetilde{\wedge}1\widetilde{\wedge}\gamma) \circ (1\widetilde{\wedge}\gamma\widetilde{\wedge}\gamma') \circ (1\widetilde{\wedge}\eta\widetilde{\wedge}1) \circ a \\ &= \mu \circ (1\widetilde{\wedge}\mu) \circ (1\widetilde{\wedge}1\widetilde{\wedge}\gamma) \circ (1\widetilde{\wedge}\eta\widetilde{\wedge}1) \circ (1\widetilde{\wedge}\gamma') \circ a \\ &= \mu \circ (1\widetilde{\wedge}\mu) \circ (1\widetilde{\wedge}\eta\widetilde{\wedge}1) \circ (1\widetilde{\wedge}\gamma) \circ (1\widetilde{\wedge}\gamma') \circ a \\ &= \mu \circ (1\widetilde{\wedge}\gamma) \circ (1\widetilde{\wedge}\gamma') \circ a \\ &= [m^{*}\Phi(a)](\gamma,\gamma').\end{aligned}$$

It suffices to check that the bottom square commutes before we complete the bottom row. We then have

$$\begin{aligned}[\tau(\Phi \otimes \Phi)(a,b)](\gamma,\gamma') &= \mu \circ (1 \widetilde{\wedge} \gamma) \circ (\mu \widetilde{\wedge} \mu) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} 1 \widetilde{\wedge} \gamma') \circ (a \widetilde{\wedge} b) \\ &= \mu \circ (1 \widetilde{\wedge} \mu) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma) \circ (\mu \widetilde{\wedge} 1) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} 1 \widetilde{\wedge} \gamma') \circ (a \widetilde{\wedge} b) \\ &= \mu \circ (1 \widetilde{\wedge} \mu) \circ (\mu \widetilde{\wedge} 1 \widetilde{\wedge} 1) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma\gamma') \circ (a \widetilde{\wedge} b) \\ &= \mu \circ (1 \widetilde{\wedge} \mu) \circ (1 \widetilde{\wedge} \mu \widetilde{\wedge} 1) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma\gamma') \circ (a \widetilde{\wedge} b) \\ &= \mu \circ (1 \widetilde{\wedge} \mu) \circ (1 \widetilde{\wedge} \gamma \widetilde{\wedge} \gamma\gamma') \circ (1 \widetilde{\wedge} \mu \widetilde{\wedge} 1) \circ (a \widetilde{\wedge} b) \\ &= [\Phi' \tau(a,b)](\gamma,\gamma'),\end{aligned}$$

completing the proof. □

Appendix A. The topology on Γ

Recall that we identified the group Γ in Theorem 4.9 as the semi-direct product $\Gamma_0 \rtimes C$, where Γ_0 is the automorphism group of the Honda formal group law over $\mathbb{F}_{p^n}$ and C is the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$, This is an identification of abstract groups; for the isomorphism $\Phi\colon E_*^\vee E \to C(\Gamma, E_*)$ to be useful, we also need to understand the topology on Γ. At the moment, we know only that this topology is profinite. The topology on Γ was described in [Str00], but many details are missing that we fill in here.

As explained in [Rav86, Lemma A2.2.16], the group Γ_0 is the group of units in the endomorphism ring D of F_n, which is a noncommutative ring obtained from the Witt ring $W\mathbb{F}_{p^n}$ by adjoining an indeterminate S subject to the relations $S^n = p$ and $Sw = w^\sigma S$, where σ denotes the generator of C. The group C acts on $W\mathbb{F}_{p^n}$ according to [Rav86, Lemma A2.2.15]. The subgroup of Γ_0 consisting of the strict isomorphisms is called the **Morava stabilizer group** S_n in [Rav86, Section 6.2].

We begin with some well-known facts about the stabilizer group S_n, whose proofs can be hard to find in the literature.

Lemma A.1. *The Morava stabilizer group S_n is a topologically finitely generated, pro-p group, and an open subgroup of finite index in $\Gamma_0 \rtimes C$.*

Here a topological group is **topologically finitely generated** if it has a dense finitely generated subgroup. Note that it follows from Lemma A.1 that $\Gamma_0 \rtimes C$, as a finite extension of S_n, is also topologically finitely generated.

Proof. First note that D itself is a profinite ring. Indeed, by Lemma A2.2.16 of [Rav86], D is a free $\mathbb{Z}_{(p)}$-module of rank n^2, and since p is central in D, we conclude that $D = \lim D/(p^k)$ as rings. It follows that Γ_0, as the group of units in D, is a profinite group (see the proof of Corollary 1.2). Hence $\Gamma_0 \rtimes C$ is a profinite group as well in the product topology.

The stabilizer group S_n is the preimage of 1 under the map of rings $D \to \mathbb{F}_{p^n}$ that takes an endomorphism of F_n to the coefficient of x in it. This map is the canonical reduction mod p on W and takes S to 0. Its kernel is the 2-sided ideal (S) generated by S, which is open since $S^n = p$. Hence this reduction map is continuous, so S_n is closed. Since $\mathbb{F}_{p^n}$ is finite, S_n has finite index in $D^\times$ and so also in $\Gamma_0 \rtimes C$, and so must also be open.

Since $S^n = p$, D is complete in the S-adic topology. Hence S_n is the inverse limit of the groups $G_k = \ker(D/S^k)^\times \to (D/S)^\times$. Since S is nilpotent in D/S^k, this kernel is $1+H_k$, where $H_k = \ker(D/S^k \to D/S)$. In particular, G_k is a finite p-group, so S_n is a pro-p group.

We now prove that S_n is topologically finitely generated. Recall from [Rav86, Lemma A2.2.16] that every element of S_n can be written as $1+\sum_{i=1}^{\infty} e_i S^i$, with $e_i \in \widetilde{\mathbb{F}_{p^n}}$, where $\widetilde{\mathbb{F}_{p^n}}$ is the set of all $e \in W\mathbb{F}_{p^n}$ such that $e^{p^n} = e$. The reduction map $W\mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ sends $\widetilde{\mathbb{F}_{p^n}}$ to $\mathbb{F}_{p^n}$ by a multiplicative bijection; $\widetilde{\mathbb{F}_{p^n}}$ is known as the set of Teichmuller lifts. Now, let

$$T = \{1+eS^k | e \in \widetilde{\mathbb{F}_{p^n}}, 1 \leq k \leq \frac{np}{p-1}\}.$$

We claim that the subgroup generated by T is dense. To see this, we first show that for all $e \in \widetilde{\mathbb{F}_{p^n}}$ and for all k, T contains an element $x_{e,k}$ such that $x_{e,k} \cong 1+eS^k \pmod{S^{k+1}}$. Indeed, this is obvious for all $k \leq \frac{np}{p-1}$. For $k > \frac{np}{p-1}$, we let $x_{e,k} = x_{e,k-n}^p$. To see that $x_{e,k-n}^p$ has the required form, we use the fact that

$$\begin{aligned}
&(1+eS^{k-n})^p \\
&= 1 + peS^{k-n} + \cdots + \binom{p}{i} e^{(p^i-1)/(p-1)} S^{i(k-n)} + \cdots + e^{(p^n-1)/(p-1)} S^{p(k-n)} \\
&\cong 1 + eS^k + e^{(p^n-1)/(p-1)} S^{p(k-n)} \pmod{S^{k+1}},
\end{aligned}$$

since $S^n = p$, and $Se = e^p S$ for $e \in \widetilde{\mathbb{F}_{p^n}}$. Now, since $k > \frac{np}{p-1}$, we see that $p(k-n) > k$, so $x_{e,k-n}^p$ is indeed a good choice for $x_{e,k}$.

Now, in order to see that T is dense, it suffices to show that for all k and $e_1, \dots, e_k \in \widetilde{\mathbb{F}_{p^n}}$, T contains an element congruent to $1+\sum_{i=1}^{k} e_i S^i$ modulo S^{k+1}. We prove this by induction on k, the base case being obvious. For the induction step, the induction hypothesis guarantees we can find an element

$$y \cong 1 + \sum_{i=1}^{k} e_i S^i + aS^{k+1} \pmod{S^{k+2}}$$

in T. We can also find an element $b \in \widetilde{\mathbb{F}_{p^n}}$ such that $a+b \cong e_{k+1} \pmod{p}$. Then

$$\begin{aligned}
yx_{b,k+1} &\cong 1 + \sum_{i=1}^{k} e_i S^i + (a+b)S^{k+1} \pmod{S^{k+2}} \\
&\cong 1 + \sum_{i=1}^{k} e_i S^i + e_{k+1} S^{k+1} \pmod{S^{k+2}},
\end{aligned}$$

as required, using the fact that $S^n = p$. □

Theorem A.2. *This isomorphism $\Gamma \cong \Gamma_0 \rtimes C$ of Theorem 4.9 is a continuous isomorphism of profinite groups. Furthermore, a subgroup of Γ is open if and only if it has finite index.*

This theorem is saying that the topology on Γ is completely determined by the group structure. This is believed to be true for a general profinite group, but remains an open question [CR02].

Proof. By [DdSMS99, Theorem 1.17], the open subgroups of a topologically finitely generated pro-p group such as S_n are precisely the subgroups of finite index. By [And76, Proposition 2], this remains true for any finite extension, such as $\Gamma_0 \rtimes C$, of such a group. It follows that the isomorphism $\Gamma_0 \rtimes C \to \Gamma$ of Theorem 4.9 is continuous. It is therefore a homeomorphism as well, since it is a map from a compact space to a Hausdorff space. □

References

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics. MR 53 #6534

[And76] Michael P. Anderson, *Subgroups of finite index in profinite groups*, Pacific J. Math. **62** (1976), no. 1, 19–28. MR 55 #8182

[Bak89] Andrew Baker, *Some families of operations in Morava K-theory*, Amer. J. Math. **111** (1989), no. 1, 95–109. MR **90d:**55030

[Bak95] ———, *A version of Landweber's filtration theorem for v_n-periodic Hopf algebroids*, Osaka J. Math. **32** (1995), no. 3, 689–699. MR **97h:**55007

[CR02] Jon M. Corson and Thomas J. Ratkovich, *On profinite groups whose power subgroups are closed*, Comm. Algebra **30** (2002), no. 9, 4189–4196. MR **2003h:**20055

[DdSMS99] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro-p groups*, second ed., Cambridge Studies in Advanced Mathematics, vol. 61, Cambridge University Press, Cambridge, 1999. MR **2000m:**20039

[DG70] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris, 1970, Avec un appendice *Corps de classes local* par Michiel Hazewinkel. MR 46 #1800

[DH04] Ethan S. Devinatz and Michael J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), 1–47.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR **97h:**55006

[GHMR03] Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk, *A resolution of the K_2-local sphere*, preprint, 2003.

[gro72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269. MR 50 #7130

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114. MR **98a:**55017

[HS99] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100. MR **99b:**55017

[HS03] ———, *Comodules and Landweber exact homology theories*, preprint, 2003.

[KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits. MR **2000a:**16031

[Lam99] T. Y. Lam, *Lectures on modules and rings*, Springer-Verlag, New York, 1999. MR **99i:**16001

[LT66] Jonathan Lubin and John Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49–59. MR 39 #214

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR **90i:**13001

[Mor85] Jack Morava, *Noetherian localisations of categories of cobordism comodules*, Ann. of Math. (2) **121** (1985), no. 1, 1–39. MR **86g:**55004

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986. MR **87j:**55003

[Rez98] Charles Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366. MR **2000i:**55023

[Str99] Neil P. Strickland, *Formal schemes and formal groups*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Amer. Math. Soc., Providence, RI, 1999, pp. 263–352. MR **2000j:**55011

[Str00] N. P. Strickland, *Gross-Hopkins duality*, Topology **39** (2000), no. 5, 1021–1033. MR **2001d:**55006

Department of Mathematics, Wesleyan University, Middletown, CT 06459
E-mail address: hovey@member.ams.org