

MONOIDAL MODEL CATEGORIES

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ABSTRACT. A monoidal model category is a model category with a closed monoidal structure which is compatible with the model structure. Given a monoidal model category, we consider the homotopy theory of modules over a given monoid and the homotopy theory of monoids. We make minimal assumptions on our model categories; our results therefore are more general, yet weaker, than the results of [10]. In particular, our results apply to the monoidal model category of topological symmetric spectra [7].

INTRODUCTION

A monoidal model category is a (closed) monoidal category that is also a model category in a compatible way. Monoidal model categories abound in nature: examples include simplicial sets, compactly generated topological spaces, and chain complexes of modules over a commutative ring. The thirty-year long search for a monoidal model category of spectra met success with the category of S -modules of [3] and the symmetric spectra of [7].

Given any monoidal category, one has categories of monoids and of modules over a given monoid. If we are working in a monoidal model category, we would like these associated categories also to be model categories, so that we can have a homotopy theory of rings and modules. The first results on this subject were obtained in [10]. This paper is a followup to that paper. In [10], the authors added the following three assumptions about a monoidal model category \mathcal{C} :

- (a) Every object of \mathcal{C} is small relative to the whole category;
- (b) \mathcal{C} satisfies the monoid axiom; and
- (c) Given a monoid A and a cofibrant left A -module M , smashing over A with M takes weak equivalences of right A -modules to weak equivalences.

The first two assumptions guarantee the existence of a model structure on the category of monoids and on the category of modules over a given monoid. The third assumption guarantees that a weak equivalence of monoids induces a Quillen equivalence of the corresponding module categories.

All these assumptions are reasonable ones in any combinatorial situation, such as simplicial sets, chain complexes, or simplicial symmetric spectra. However, for any category of topological spaces the first and third assumption will fail. Furthermore, in the category of topological symmetric spectra the second assumption is not known to hold, though there is recent unpublished work of May and coauthors on his subject.

The goal of this paper, then, is to investigate what can be said when these assumptions do not hold. After a preliminary section reminding the reader of some

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basic definitions and facts about model categories, we begin in the second section by showing that one always gets a model category of modules over a cofibrant monoid. Furthermore, under a minor assumption on our model category \mathcal{C} , we show that a weak equivalence of cofibrant monoids induces a Quillen equivalence of the corresponding module categories. Also, a Quillen equivalence F of monoidal model categories induces a Quillen equivalence between R -modules and FR -modules, for R a monoid which is cofibrant in the domain of F . In the third section, we show that, if the unit S in \mathcal{C} is cofibrant, then, though we do not get a model category of monoids in general, we do at least get a homotopy category of monoids. In particular, given a general monoid A , we can find a cofibrant monoid QA and a weak equivalence and homomorphism $QA \rightarrow A$. Then the model category $QA\text{-mod}$ is the homotopy invariant replacement for the category $A\text{-mod}$, which may not even be a model category. We also show that the homotopy category of monoids is itself homotopy invariant. In particular, there is a homotopy category of monoids of topological symmetric spectra, and this homotopy category is equivalent to the homotopy category of monoids of simplicial symmetric spectra.

The most obvious question left unaddressed in this paper concerns the category of *commutative* monoids in a symmetric monoidal model category. What do we need to know to get a model structure on commutative monoids? Can we get a homotopy category of commutative monoids in any symmetric monoidal model category? The author does not know the answer to these questions.

The author would like to thank his coauthors Brooke Shipley and Jeff Smith. This paper grew out of [7], when the authors of that paper realized that topological spaces are not as simple as they had originally thought. The author would also like to thank Gaunce Lewis and Peter May for helping him come to that realization, which of course they have understood for years.

1. MODULES

In this section we investigate model categories of modules over a monoid A in a cofibrantly generated monoidal model category \mathcal{C} . For cofibrantly generated model categories, and for all terms left undefined here, we refer the reader to [5]; cofibrantly generated model categories appear in Section 2.1.

Monoidal model categories are discussed in [5, Chapter 4], but we will remind the reader of the basic definition. In a monoidal category \mathcal{C} , we will denote the monoidal product by \wedge and the unit by S . Note that in model category theory, functions seem to come in adjoint pairs. We will therefore consider a *closed* monoidal category rather than a general monoidal category. This means that both functors $X \wedge -$ and $- \wedge X$ have right adjoints natural in X . For our purposes, the closed structure just guarantees for us that both functors $X \wedge -$ and $- \wedge X$ preserve colimits.

Definition 1.1. Suppose \mathcal{C} is a closed monoidal category. Given maps $f: A \rightarrow B$ and $g: X \rightarrow Y$ in \mathcal{C} , define the *pushout smash product* $f \square g$ of f and g to be the map $(A \wedge Y) \amalg_{A \wedge X} (B \wedge X) \rightarrow B \wedge Y$.

Definition 1.2. Suppose \mathcal{C} is a closed monoidal category which is also a model category. Then \mathcal{C} is a *monoidal model category* if the following conditions hold.

- (a) If f and g are cofibrations, so is $f \square g$. If one of f or g is in addition a weak equivalence, so is $f \square g$.

- (b) Both maps $q \wedge X: QS \wedge X \rightarrow S \wedge X \cong X$ and $X \wedge q: X \wedge QS \rightarrow X$ are weak equivalences for all cofibrant X . Here the map $QS \rightarrow S$ is a cofibrant replacement for S .

The second condition is a consequence of the first in case the unit S is cofibrant. This is usually the case, but S is not cofibrant in the category of S -modules of [3]. Without the second condition, the homotopy category of a monoidal model category would not be a monoidal category, because there would not be a unit. With it, it is an exercise in derived functors, carried out in [5, Chapter 4], to verify that the homotopy category is indeed a monoidal category.

Of course, we only need half the second condition in case \mathcal{C} is symmetric monoidal, as it usually is in our examples.

We point out, following the insight of Stefan Schwede, that the second condition in Definition 1.2 is equivalent to requiring that both maps $X \rightarrow \text{Hom}_\ell(QS, X)$ and $X \rightarrow \text{Hom}_r(QS, X)$ are weak equivalences for all fibrant X , where Hom_ℓ and Hom_r are the two adjoints that define the closed structure on \mathcal{C} . See [5, Lemma 4.2.7].

Examples of symmetric monoidal model categories are discussed in [5, Section 4.2], and include the categories of simplicial sets, compactly generated topological spaces, S -modules [3], symmetric spectra [7], and topological symmetric spectra.

The reader should note that, in a monoidal model category, smashing with a cofibrant object preserves cofibrations and trivial cofibrations. It follows from Ken Brown's lemma [5, Lemma 1.1.12] that smashing with a cofibrant object also preserves weak equivalences between cofibrant objects.

Before stating the basic theorem about model categories of modules, we recall that, given a set of maps I , I -cell is the class of all transfinite compositions of pushouts of I [5, Section 2.1]. There are related definitions of I -inj and I -cof to be found in [5, Section 2.1]. Also, an object X is called *small* relative to a class of maps \mathcal{D} if maps from X into a long enough transfinite composition of maps of \mathcal{D} must factor through some stage of the colimit [5, Section 2.1].

Theorem 1.3. *Suppose \mathcal{C} is a cofibrantly generated monoidal model category with generating cofibrations I and generating trivial cofibrations J . Let A be a monoid in \mathcal{C} , and suppose the following conditions hold.*

- (1) *The domains of I are small relative to $(A \wedge I)$ -cell.*
- (2) *The domains of J are small relative to $(A \wedge J)$ -cell.*
- (3) *Every map of $(A \wedge J)$ -cell is a weak equivalence.*

Then there is a cofibrantly generated model structure on the category of left A -modules, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in \mathcal{C} .

There is an obvious analogue of this theorem for right A -modules.

Proof. By adjointness, the fibrations of A -modules form the class $(A \wedge J)$ -inj, and the trivial fibrations form the class $(A \wedge I)$ -inj. We therefore define the cofibrations of A -modules to be the class $(A \wedge I)$ -cof, and take our generating trivial cofibrations to be $A \wedge J$. Since each element of J is in I -cof, each element of $A \wedge J$ is in $(A \wedge I)$ -cof, and so the maps of $(A \wedge J)$ -cof are cofibrations in A -mod.

The category of A -modules is certainly bicomplete, with limits and colimits taken in \mathcal{C} . The retract and two out of three axioms are immediate, as is the lifting

axiom for cofibrations and trivial fibrations. By assumption, the domains of I are small relative to $(A \wedge I)$ -cell. By adjointness, it follows that the domains of $A \wedge I$ are small in A -mod relative to $(A \wedge I)$ -cell. Thus the small object argument [5, Theorem 2.1.14] gives us the cofibration-trivial fibration half of the factorization axiom. Similarly, the domains of $A \wedge J$ are small in A -mod relative to $(A \wedge J)$ -cell. We can then factor any map in A -mod into a map of $(A \wedge J)$ -cell followed by a fibration. We have already seen that the maps of $(A \wedge J)$ -cell are cofibrations, and by assumption they are weak equivalences. This gives the other half of the factorization axiom.

For the remaining lifting axiom, suppose f is a cofibration and weak equivalence. Factor $f = pi$, where $i \in (A \wedge J)$ -cell and p is a fibration of A -modules. Since i is a weak equivalence, and so is f , it follows that p is a weak equivalence. Thus f has the left lifting property with respect to p . By the Retract Argument [5, Lemma 1.1.9], f is a retract of i , and so has the left lifting property with respect to all fibrations of A -modules, as required. \square

Corollary 1.4. *Suppose \mathcal{C} is a cofibrantly generated monoidal model category, and suppose A is a monoid which is cofibrant in \mathcal{C} . Then there is a cofibrantly generated model structure on the category of (left, or right) A -modules, where a map is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in \mathcal{C} . Furthermore, a cofibration of A -modules is a cofibration in \mathcal{C} .*

Proof. For definiteness, we work with left A -modules. Since A is cofibrant in \mathcal{C} , every map of $A \wedge I$ is a cofibration, so $(A \wedge I)$ -cell $\subseteq I$ -cof. Since the domains of I are small relative to I -cof, they are certainly small relative to $(A \wedge I)$ -cell. Similarly, $(A \wedge J)$ -cell $\subseteq J$ -cof, so the domains of J are small relative to $(A \wedge J)$ -cell, and the maps of $(A \wedge J)$ -cell are weak equivalences. \square

As an example, we would like to consider the model category of topological spaces. This is not, however, a monoidal model category, since the functor $X \times -$ need not have a right adjoint unless X is locally compact Hausdorff. To get around this, it is usual to consider some version of compactly generated spaces. We use the definitions of [8, Appendix A], the properties of which are summarized in [5, Section 2.4]. In particular, a subset U of a topological space X is *compactly open* if, for every continuous $f: K \rightarrow X$, where K is compact Hausdorff, the preimage $f^{-1}(U)$ is open. The space X is called a *k-space* if every compactly open space is open. A space is called *compactly generated* if it is both a *k-space* and weak Hausdorff; *i.e.*, for every continuous $f: K \rightarrow X$ where K is compact Hausdorff, the image $f(K)$ is closed. Then both the category \mathcal{K} of *k-spaces* and the category \mathcal{T} of compactly generated spaces are cofibrantly generated symmetric monoidal model categories [5, Section 4.2].

Recall that, given a cofibrantly generated model category \mathcal{C} with generating trivial cofibrations J , the *monoid axiom* [10] asserts that every element of $(\mathcal{C} \wedge J)$ -cell is a weak equivalence.

Lemma 1.5. *The model category \mathcal{T} satisfies the monoid axiom. Furthermore, the maps of $(\mathcal{T} \wedge J)$ -cell are closed inclusions.*

Proof. The generating trivial cofibrations J for \mathcal{T} can be taken to be the inclusions $I^n \times \{0\} \rightarrow I^n \times I$, where I is the unit interval. In particular, they are closed inclusions of strong deformation retracts. One can then easily verify that every

map of $A \wedge J$ is again a closed inclusion of a strong deformation retract (using [8, Appendix A] for the closed inclusion part). Hence, every pushout of $A \wedge J$ is again a closed inclusion of a strong deformation retract, so, in particular, a closed inclusion and weak equivalence. Now, in \mathcal{T} , compact spaces are finite [5, Section 2.1] with respect to closed inclusions. It follows from [5, Corollary 7.4.2] that every map of $(\mathcal{C} \wedge J)$ -cell is a weak equivalence (and a closed inclusion), as required. \square

In particular, Theorem 1.3 applies, and we get a model category of modules over any monoid in \mathcal{T} .

We do not know if the monoid axiom holds in the model category \mathcal{K} of k -spaces. The argument above fails, because compact spaces are not finite in \mathcal{K} relative to closed inclusions, but only relative to closed T_1 inclusions. Here a closed inclusion $f: X \rightarrow Y$ is a closed T_1 inclusion if every point of Y not in X is closed. The maps of $A \wedge J$ will not be closed T_1 inclusions if A is not itself T_1 .

In order for the model category of A -modules to be useful, it must be both homotopy invariant in appropriate senses and have good properties. We begin with the homotopy invariance, which requires the basic notions of Quillen functors and Quillen equivalences [5, Section 1.3].

Theorem 1.6. *Suppose \mathcal{C} is a cofibrantly generated monoidal model category such that the domains of the generating cofibrations can be taken to be cofibrant. Suppose $f: A \rightarrow A'$ is a weak equivalence of monoids that are cofibrant in \mathcal{C} . Then the induction functor induced by f and its right adjoint, the restriction functor, define a Quillen equivalence from the model category of left (resp. right) A -modules to the model category of left (resp. right) A' -modules.*

Proof. Again, we work with left modules for definiteness. In this case the induction functor takes M to $A' \wedge_A M$. The restriction functor obviously preserves weak equivalences and fibrations, so is a right Quillen functor. Furthermore, the restriction functor reflects weak equivalences as well. It follows from [5, Corollary 1.3.16] that induction is a Quillen equivalence if and only if for all cofibrant A -modules M , the map $M \rightarrow A' \wedge_A M$ is a weak equivalence. Because the category of A -modules is cofibrantly generated, we may as well assume that M is the colimit of a colimit-preserving functor $\alpha \rightarrow A\text{-mod}$, where α is an ordinal, $M_0 = 0$ and each map $M_\beta \rightarrow M_{\beta+1}$ is a pushout of a map of $A \wedge I$.

We will prove by transfinite induction that the map $i_\beta: M_\beta \rightarrow A' \wedge_A M_\beta$ is a weak equivalence for all $\beta \leq \alpha$, taking $M_\alpha = \text{colim}_{\alpha < \beta} M_\beta = M$. Getting the induction started is easy, since $M_0 = 0$. For the successor ordinal case, suppose that i_β is a weak equivalence. We have a pushout diagram

$$\begin{array}{ccc} A \wedge K & \longrightarrow & A \wedge L \\ \downarrow & & \downarrow \\ M_\beta & \longrightarrow & M_{\beta+1} \end{array}$$

where $K \rightarrow L$ is some map of I . Both horizontal maps are cofibrations in \mathcal{C} . Furthermore, because K and L are by assumption cofibrant in \mathcal{C} , each object in the diagram is cofibrant in \mathcal{C} . By applying the functor $A' \wedge_A -$, we get an analogous

pushout diagram

$$\begin{array}{ccc} A' \wedge K & \longrightarrow & A' \wedge L \\ \downarrow & & \downarrow \\ A' \wedge_A M_\beta & \longrightarrow & A' \wedge_A M_{\beta+1} \end{array}$$

where again the horizontal maps are cofibrations in \mathcal{C} , and each object is cofibrant in \mathcal{C} . There is a map from the first pushout square to the second, which by the induction hypothesis is a weak equivalence on the lower left square. Since K is cofibrant, smashing with K preserves trivial cofibrations in \mathcal{C} , and hence, by Ken Brown's lemma [5, Lemma 1.1.12], preserves weak equivalences between cofibrant objects of \mathcal{C} . Thus the map of pushout squares is also a weak equivalence on the upper left corner, and by similar reasoning, on the upper right corner. Dan Kan's cubes lemma [5, Lemma 5.2.6] then shows that it is a weak equivalence on the lower right corner as well. Thus $i_{\beta+1}$ is a weak equivalence.

Now consider the limit ordinal case. Here we assume i_γ is a weak equivalence for all $\gamma < \beta$. We then have a map of sequences

$$\begin{array}{ccccccc} M_0 & \longrightarrow & M_1 & \longrightarrow & \dots & \longrightarrow & M_\gamma & \longrightarrow & \dots \\ i_0 \downarrow & & i_1 \downarrow & & & & i_\gamma \downarrow & & \\ A' \wedge_A M_0 & \longrightarrow & A' \wedge_A M_1 & \longrightarrow & \dots & \longrightarrow & A' \wedge_A M_\gamma & \longrightarrow & \dots \end{array}$$

where the vertical maps are all weak equivalences, and the horizontal maps are cofibrations of cofibrant objects. Then [4, Proposition 18.4.1] implies that i_β is also a weak equivalence, as required. \square

The author learned the following example from Neil Strickland.

Example 1.7. Let us consider the category \mathcal{T}_* of compactly generated pointed spaces. This is a monoidal model category under the smash product, and the same argument used in Lemma 1.5 will prove that the monoid axiom holds in \mathcal{T}_* , and that there is a model category of modules over any monoid. Let A denote the nonnegative natural numbers together with infinity, given the discrete topology. With infinity as the base point, A is a monoid in \mathcal{T}_* , with unit 0. Let A' be the same set as A , but with the one-point compactification topology. Then A' is also a topological monoid, with basepoint at infinity and unit 0, and the identity $A \xrightarrow{f} A'$ is a homomorphism and weak equivalence of monoids. Note that A' is not cofibrant as a topological space, so Theorem 1.6 does not apply. In fact, the induction functor does not define a Quillen equivalence from A -modules to A' -modules. Indeed, take $M = A \wedge S^1$. Then the map $M \rightarrow A' \wedge_A M$ is just the suspension of f . The suspension of A is an infinite wedge of circles, but the suspension of A' is the Hawaiian earring. In particular, $f \wedge S^1$ is not a weak equivalence, so induction can not be a Quillen equivalence.

In light of this example, it seems to the author that one should avoid considering modules over monoids which are not cofibrant in \mathcal{C} , just we generally avoid suspending non-cofibrant topological spaces. There are some categories, however, where a weak equivalence of monoids always induces a Quillen equivalence of the corresponding module categories. This is true in S -modules [3, Theorem 3.8] and in simplicial symmetric spectra [7, Theorem 5.5.9]. See [10, Theorem 3.3].

Theorem 1.6 shows that the model category of A -modules is homotopy invariant under weak equivalences of cofibrant objects in \mathcal{C} . But we would also like the model category of A -modules to be homotopy invariant under weak equivalences of monoidal model categories \mathcal{C} . Recall that a (left) Quillen functor F between monoidal model categories is called a *monoidal Quillen functor* if it is (strictly) monoidal and the induced map $F(QS) \rightarrow F(S) \cong S$ is a weak equivalence. This second condition is easy to overlook; it is essential in case the unit S of \mathcal{C} is not cofibrant in order to be sure LF is a monoidal functor.

Theorem 1.8. *Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal Quillen equivalence of cofibrantly generated monoidal model categories, with right adjoint U . Suppose that A is a monoid in \mathcal{C} which is cofibrant. Then F induces a Quillen equivalence $F: A\text{-mod} \rightarrow FA\text{-mod}$.*

Proof. Since F is monoidal, FA is a monoid in \mathcal{D} ; since F preserves cofibrant objects, FA is cofibrant in \mathcal{D} . Given an A -module M , FM is an FA -module with structure map $FA \wedge FM \cong F(A \wedge M) \rightarrow FM$. Hence F does define a functor $F: A\text{-mod} \rightarrow FA\text{-mod}$. Let $\eta: X \rightarrow UFX$ denote the unit of the adjunction, and let $\varepsilon: FUX \rightarrow X$ denote the counit. Given an FA -module N , UN is an A -module; its structure map is given by the composite

$$A \wedge UN \xrightarrow{\eta \wedge 1} UFA \wedge UN \rightarrow U(FA \wedge N) \rightarrow UN$$

where the second map is adjoint to the composite

$$F(UFA \wedge UN) \xrightarrow{\cong} FUF A \wedge FUN \xrightarrow{\varepsilon \wedge \varepsilon} FA \wedge N.$$

One can easily check that the resulting functor $U: FA\text{-mod} \rightarrow A\text{-mod}$ is right adjoint to F . The functor U clearly preserves fibrations and trivial fibrations, so is a right Quillen functor. Furthermore, U reflects weak equivalences between fibrant objects, since U does so as a functor from \mathcal{D} to \mathcal{C} . Using [5, Corollary 1.3.16], we need only check that the map $X \rightarrow ULFX$ is a weak equivalence for cofibrant A -modules X . Here L is a fibrant replacement functor in $FA\text{-mod}$, not the fibrant replacement functor R in \mathcal{D} . Nevertheless, there is a weak equivalence $FX \rightarrow LFX$, and since LFX is fibrant in \mathcal{D} , there is a map $RF X \rightarrow LFX$ in \mathcal{D} which is necessarily a weak equivalence. Thus the map $URFX \rightarrow ULFX$ is a weak equivalence in \mathcal{C} . Since X is a cofibrant A -module, and in particular cofibrant in \mathcal{C} , the map $X \rightarrow URFX$ is a weak equivalence. It follows that the map $X \rightarrow ULFX$ is a weak equivalence as desired. \square

We now discuss some of the properties of the model category $A\text{-mod}$. If \mathcal{C} is a symmetric monoidal category, and A is a commutative monoid, then it is well known that $A\text{-mod}$ is also a symmetric monoidal category. We would like the model structure on $A\text{-mod}$ to be compatible with this symmetric monoidal structure.

Proposition 1.9. *Suppose \mathcal{C} is a cofibrantly generated symmetric monoidal model category. In addition, suppose that either*

- (1) *The unit S is cofibrant and A is a commutative monoid satisfying the conditions of Theorem 1.3; or*
- (2) *A is a commutative monoid cofibrant in \mathcal{C} .*

Then the model category $A\text{-mod}$ is a cofibrantly generated symmetric monoidal model category.

Proof. It is well-known that $A\text{-mod}$ is closed symmetric monoidal: see [7, Section 2.2] for details. The symmetric monoidal structure is denoted \wedge_A , and we then have an analogous definition of \square_A . Let I be the set of generating cofibrations of \mathcal{C} and let J be the set of generating trivial cofibrations. Then $A \wedge I$ is the set of generating cofibrations of $A\text{-mod}$, and $A \wedge J$ is the set of generating trivial cofibrations. We have

$$(A \wedge I) \square_A (A \wedge I) = (A \wedge I) \square I = A \wedge (I \square I) \subseteq A \wedge (I\text{-cof}) \subseteq (A \wedge I)\text{-cof}.$$

Thus $f \square_A g$ is a cofibration of A -modules if f and g are, using [10, Lemma 2.3]. See also [7, Corollary 5.3.5]. A similar argument shows that $f \square_A g$ is a trivial cofibration if either f or g is. Thus the pushout product part of the definition of a monoidal model category holds. In the first case, we are done; since S is cofibrant in \mathcal{C} , A is cofibrant in $A\text{-mod}$. In the second case, let QS be a cofibrant replacement for the unit S in \mathcal{C} , so that QS is cofibrant and we have a weak equivalence $QS \rightarrow S$. Then $A \wedge QS$ is cofibrant in $A\text{-mod}$, and, since A is cofibrant and \mathcal{C} is monoidal, the A -module map $A \wedge QS \rightarrow A$ is a weak equivalence. Thus $A \wedge QS$ is a cofibrant replacement for the unit A in $A\text{-mod}$. We must show that, if M is a cofibrant A -module, then the map $(A \wedge QS) \wedge_A M \rightarrow A \wedge_A M = M$ is still a weak equivalence. But $(A \wedge QS) \wedge_A M \cong QS \wedge M$. Since M is cofibrant in \mathcal{C} , the desired result holds. \square

Note that the Quillen equivalences of Theorems 1.6 and 1.8 are monoidal Quillen equivalences in case the monoids involved are commutative.

In case A is not commutative, there is still a closed action of \mathcal{C} on $A\text{-mod}$; $M \wedge X$ is an A -module if M is an A -module and X is arbitrary. This action also respects the model structures: $f \square g$ is a cofibration of A -modules if f is a cofibration of A -modules and g is a cofibration. Furthermore, $f \square g$ is a weak equivalence if either f or g is. However, we will have trouble with the unit unless we assume either A or S is cofibrant, just as above.

Since Hirschhorn's landmark treatment [4], it has become clear that the right collection of model categories to work with is the collection of left proper cellular model categories. Hirschhorn shows that one can perform Bousfield localization in this setting.

Proposition 1.10. *Suppose \mathcal{C} is a left proper cellular monoidal model category. Suppose A is a monoid which is cofibrant in \mathcal{C} . Then the model category $A\text{-mod}$ is also left proper and cellular.*

Because the definition of cellular is technical, the proof of this proposition would take us too far afield. It can be proved by the methods of [6, Section 6].

2. ALGEBRAS

In this section we study the category $A\text{-alg}$ of algebras over a commutative monoid A in a cofibrantly generated symmetric monoidal model category \mathcal{C} . Note that in case $A = S$, an S -algebra is the same thing as a monoid in \mathcal{C} . Furthermore, an A -algebra is just a monoid in the symmetric monoidal category $A\text{-mod}$. We use this to reduce to the case of monoids.

The obvious definitions to make for a model structure on $A\text{-alg}$ are the following. We define a homomorphism $f: X \rightarrow Y$ of A -algebras to be a *weak equivalence (fibration)* if and only if f is a weak equivalence (fibration) in \mathcal{C} . Then define f

to be a *cofibration* if and only if f has the left lifting property with respect to all homomorphisms of A -algebras which are both weak equivalences and fibrations.

Note that the forgetful functor $A\text{-alg} \rightarrow \mathcal{C}$ has a left adjoint, the free algebra functor T . Of course, $T(X) = A \wedge \coprod_{n \geq 0} X^{\wedge n}$, where $X^{\wedge 0} = S$. We will usually use this only when $A = S$.

We begin by slightly generalizing the main result of [10].

Theorem 2.1. *Suppose \mathcal{C} is a cofibrantly generated symmetric monoidal model category, with generating cofibrations I and generating trivial cofibrations J . Suppose the following conditions hold.*

- (1) *The domains of I are small relative to $(\mathcal{C} \wedge I)$ -cell.*
- (2) *The domains of J are small relative to $(\mathcal{C} \wedge J)$ -cell.*
- (3) *The maps of $(\mathcal{C} \wedge J)$ -cell are weak equivalences; i.e., the monoid axiom holds.*

Let A be a commutative monoid in \mathcal{C} . Then $A\text{-alg}$ is a cofibrantly generated model category where a map of A -algebras is a weak equivalence or fibration if and only if it is so in \mathcal{C} .

Proof. We first point out that we can assume that $A = S$. Indeed, our conditions guarantee that $A\text{-mod}$ is a cofibrantly generated model category (see Theorem 1.3) which is symmetric monoidal, and that the pushout product half of the definition of a monoidal model category holds (see the proof of Proposition 1.9). Furthermore, we have $A\text{-mod} \wedge_A (A \wedge I) = A\text{-mod} \wedge I \subseteq \mathcal{C} \wedge I$. Thus the domains of I are small in \mathcal{C} relative to $(A\text{-mod} \wedge_A (A \wedge I))$ -cell, and so the domains of $A \wedge I$ are small in $A\text{-mod}$ relative to $(A\text{-mod} \wedge_A (A \wedge I))$ -cell. Similarly, the domains of $A \wedge J$ are small in $A\text{-mod}$ relative to $(A\text{-mod} \wedge_A (A \wedge J))$ -cell. And the maps of $(A\text{-mod} \wedge_A (A \wedge J))$ -cell are in particular maps of $(\mathcal{C} \wedge J)$ -cell, so are weak equivalences. Thus the category $A\text{-mod}$ satisfies the same conditions as does \mathcal{C} (except for the second half of the definition of a monoidal model category), and so we may as well assume that $A = S$.

It is well-known that the category $S\text{-alg}$ is bicomplete, and the two out of three and retract axioms are immediate consequences of the definitions. The cofibration-trivial fibration half of the lifting axiom is also immediate. For the factorization axioms, we use the sets $T(I)$ and $T(J)$. Adjointness guarantees that the cofibrations in $S\text{-alg}$ are the elements of $T(I)\text{-cof}$, and the trivial fibrations are the elements of $T(I)\text{-inj}$. Similarly, the fibrations are the elements of $T(J)\text{-inj}$. To understand the maps of $T(I)\text{-cell}$, we need to understand the pushout in $S\text{-alg}$ of a map $T(g): T(K) \rightarrow T(L)$ through a map $T(K) \rightarrow X$. This pushout is described in [10, Lemma 5.2] as a countable composition of maps $P_i \xrightarrow{h_i} P_{i+1}$. Each map h_i is a pushout in \mathcal{C} of $X^{\wedge(i+1)} \wedge g_i$, where g_i is very similar to the inclusion of the fat wedge into the product. That is, the target of g_i is $L^{\wedge(i)}$, and the source is analogous to the subset of the smash product where at least one term is in K . In any case, one can see from the fact that \mathcal{C} is monoidal that g_i is a cofibration if $g \in I$, and is a trivial cofibration if $g \in J$. It follows that the maps of $T(I)\text{-cell}$ are in $(\mathcal{C} \wedge I\text{-cof})\text{-cell}$, which is contained in $(\mathcal{C} \wedge I)\text{-cof}$. Since the domains of I are small relative to $(\mathcal{C} \wedge I)\text{-cell}$, they are also small relative to $(\mathcal{C} \wedge I)\text{-cof}$ by [5, Proposition 2.1.16]. Thus, by adjointness, the domains of $T(I)$ are small in $S\text{-alg}$ relative to $T(I)\text{-cell}$. The small object argument then gives the desired factorization into a cofibration followed by a trivial fibration.

A very similar argument shows that the domains of J are small relative to $T(J)$ -cell, and also that the maps of $T(J)$ -cell, since they are in $(\mathcal{C} \wedge J)$ -cof, are weak equivalences. Thus the small object argument applied to $T(J)$ gives the desired factorization into a trivial cofibration followed by a fibration. The proof that trivial cofibrations have the left lifting property with respect to fibrations then uses the retract argument, as in the proof of Theorem 1.3. \square

Example 2.2. For an example where Theorem 2.1 applies but the simpler version of [10] does not, let \mathcal{C} be the category of compactly generated spaces. We have already seen that the monoid axiom holds here, (see Lemma 1.5). One can also verify that the elements of $(\mathcal{C} \wedge I)$ -cell are closed inclusions, using the results of [8, Appendix A]. It follows that the domains of I are small (in fact, finite) relative to $(\mathcal{C} \wedge I)$ -cell, and also that the domains of J are small relative to $(\mathcal{C} \wedge J)$ -cell. Thus we do get a model category of monoids of compactly generated spaces.

It is interesting to note that, in the situation of Theorem 2.1, the category A -alg is a model category even though the category A -mod may not be a monoidal model category unless either A or S is cofibrant.

In any case, we are really interested in the case where the hypotheses of Theorem 2.1 do not hold. In this case, we obtain the following result. There is no question that this result is unsatisfactory, but it seems to be the best we can do.

Theorem 2.3. *Suppose \mathcal{C} is a cofibrantly generated symmetric monoidal model category. Suppose that A is either S or a commutative monoid which is cofibrant in \mathcal{C} . Then the category A -alg is almost a model category, in the following precise sense.*

- (1) *A -alg is bicomplete and the two out of three and retract axioms hold.*
- (2) *Cofibrations have the left lifting property with respect to trivial fibrations, and trivial cofibrations whose source is cofibrant in A -mod have the left lifting property with respect to fibrations.*
- (3) *Every map whose source is cofibrant in A -mod can be functorially factored into a cofibration followed by a trivial fibration, and also can be functorially factored into a trivial cofibration followed by a fibration.*

Furthermore, cofibrations whose source is cofibrant in A -mod are cofibrations in A -mod, and fibrations and trivial fibrations are closed under pullbacks.

Proof. The category A -alg is the category of monoids in A -mod, which itself is a cofibrantly generated monoidal model category, by Proposition 1.9. Thus we can assume that $A = S$. We have already seen that bicompleteness and the retract and two out of three axioms hold. The lifting axiom for cofibrations and trivial fibrations holds by definition. As before, adjointness implies that the trivial fibrations form the class $T(I)$ -inj, so the cofibrations form the class $T(I)$ -cof. The fibrations form the class $T(J)$ -inj, so the elements of $T(J)$ -cof have the left lifting property with respect to fibrations. Recall from the proof of Theorem 2.1 that the pushout in S -alg of a map $T(K) \xrightarrow{T(g)} T(L)$ through a map $T(K) \rightarrow X$ of monoids is a countable composition of maps $P_i \xrightarrow{f_i} P_{i+1}$, where f_i is the pushout in \mathcal{C} of a map $X^{\wedge(i+1)} \wedge g_i$. The map g_i is, as we have said, a cofibration if g is so, and a trivial cofibration if g is so. Thus, if X is cofibrant in \mathcal{C} , a pushout in $\mathcal{M}(\mathcal{C})$ of a map of $T(I)$ through a map to X is a cofibration in \mathcal{C} and a pushout of a map of $T(J)$ through a map to X is a trivial cofibration in \mathcal{C} . Since the domains of I are small relative to cofibrations in

\mathcal{C} , it follows by adjointness that the domains of $T(I)$ are small relative to transfinite compositions of pushouts of maps of $T(I)$ as long as the initial stage is cofibrant in \mathcal{C} . The small object argument then allows us to functorially factor any map in $S\text{-alg}$ whose source is cofibrant in \mathcal{C} into a cofibration followed by a trivial fibration. A similar argument, with $T(J)$ replacing $T(I)$, allows us to functorially factor any map in $S\text{-alg}$ whose source is cofibrant in \mathcal{C} into a trivial cofibration followed by a fibration.

Now, given any cofibration f in $\mathcal{M}(\mathcal{C})$ whose source is cofibrant, we can factor $f = pi$, where i is a transfinite composition of pushouts of maps of $T(I)$, and is therefore a cofibration in \mathcal{C} , and p is a trivial fibration. The retract argument shows that f is a retract of i , and so is also a cofibration in \mathcal{C} . Now suppose f is a trivial cofibration, again with source cofibrant in \mathcal{C} . Then we can factor $f = qj$, where q is a fibration of monoids and j is a transfinite composition of pushouts of $T(J)$. Since j is also a weak equivalence, so is q . Thus f has the left lifting property with respect to q , and so f is a retract of j . Then f has the left lifting property with respect to fibrations, as desired. \square

We should point out that it is not even clear that the cofibrations $T(J)$ are weak equivalences in $\mathcal{M}(\mathcal{C})$. This will be true if we can choose the domains of J to be cofibrant.

The information contained in Theorem 2.3 is enough to carry out most of the usual constructions in model category theory, at least as long as we have some element of $A\text{-alg}$ which is cofibrant in $A\text{-mod}$. For this, we must require that the unit S is cofibrant in \mathcal{C} . Then the initial object A of $A\text{-alg}$ is cofibrant in $A\text{-mod}$. Hence there is a cofibrant replacement functor Q in $A\text{-alg}$ and a natural trivial fibration $QX \rightarrow X$. There is a fibrant replacement functor R defined on $A\text{-algebras}$ which are cofibrant in $A\text{-mod}$, in particular on cofibrant $A\text{-algebras}$, and a natural trivial cofibration $X \rightarrow RX$. In particular, though the composite functor QR does not make sense in general, the composite RQ does, and we find that the quotient category of $A\text{-alg}$ obtained by inverting the weak equivalences is equivalent to the quotient category of the cofibrant and fibrant objects.

One does have to be careful though. We can not recognize trivial cofibrations in $A\text{-alg}$ as the class of maps with the left lifting property with respect to fibrations. But a map in $A\text{-alg}$ whose source is cofibrant in $A\text{-mod}$ is a trivial cofibration if and only if it has the left lifting property with respect to all fibrations. Similarly, one does not know that a pushout of a trivial cofibration f in $A\text{-alg}$ through a map g is a trivial cofibration unless both the source of f and the target of g are cofibrant in $A\text{-mod}$. Cylinder and path objects need not exist unless the object in question is cofibrant in $A\text{-mod}$.

Nevertheless, we can follow the usual definitions in the standard construction of the homotopy category of a model category. See for example [5, Chapter 1], [2], or [4, Chapters 8 and 9]. In order for the notion of left homotopy to have any content, one must assume that the sources of one's maps are cofibrant in $A\text{-mod}$. Similarly, for right homotopy, one must assume that the target is cofibrant in $A\text{-mod}$. It is not clear that right homotopy is an equivalence relation if the target is cofibrant in $A\text{-mod}$ and fibrant, as one would expect. One must first prove that the notions of left and right homotopy coincide if the source is cofibrant and the target is cofibrant and fibrant. Then, since left homotopy is an equivalence relation in that situation, so is right homotopy.

In the end, we obtain the following theorem.

Theorem 2.4. *Suppose \mathcal{C} is a cofibrantly generated symmetric monoidal model category, where the unit S is cofibrant. Suppose A is a commutative monoid which is cofibrant in \mathcal{C} . Let $\text{Ho } A\text{-alg}$ denote the category obtained from $A\text{-alg}$ by formally inverting the weak equivalences. Then there is an equivalence of categories $(A\text{-alg})_{cf}/\sim \rightarrow \text{Ho } A\text{-alg}$, where $(A\text{-alg})_{cf}$ is the full subcategory of cofibrant and fibrant A -algebras, and \sim denotes the homotopy equivalence relation. In particular, $\text{Ho } A\text{-alg}$ exists. A map in $A\text{-alg}$ is an isomorphism in $\text{Ho } A\text{-alg}$ if and only if it is a weak equivalence.*

We now show that the homotopy category of monoids is homotopy invariant. The following theorem is analogous to Theorem 1.6.

Theorem 2.5. *Suppose \mathcal{C} is a cofibrantly generated symmetric monoidal model category such that the unit S is cofibrant and the domains of the generating cofibrations can be taken to be cofibrant. Suppose $f: A \rightarrow A'$ is a weak equivalence of commutative monoids which are cofibrant in \mathcal{C} . Then the derived functors of induction and restriction define an adjoint equivalence of categories $\text{Ho } A\text{-alg} \rightarrow \text{Ho } A'\text{-alg}$.*

Proof. One can easily check that induction which takes M to $A' \wedge_A M$, defines a functor from A -algebras to A' -algebras, which is left adjoint to the restriction functor. The restriction functor obviously preserves fibrations and trivial fibrations, and reflects weak equivalences between fibrant objects. It follows that induction preserves cofibrations, and those trivial cofibrations whose source is cofibrant in $A\text{-mod}$. One can then easily check that the functor which takes M to $A' \wedge_A QM$ is the total left derived functor of induction, and that its right adjoint is the functor which takes M to RQM . The situation is very similar to the usual Quillen functor formalism, it is just that we need to apply Q before applying the fibrant replacement functor R since R is not globally defined.

The same argument as in [5, Corollary 1.3.16] applies, so we are reduced to showing that the map $M \rightarrow A' \wedge_A M$ is a weak equivalence for all cofibrant A -algebras M . But since every cofibrant A -algebra is also a cofibrant A -module, Theorem 1.6 completes the proof. \square

Then the following theorem is analogous to Theorem 1.8.

Theorem 2.6. *Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal Quillen equivalence of cofibrantly generated symmetric monoidal model categories. Suppose as well that S is cofibrant in \mathcal{C} , and A is a commutative monoid which is cofibrant in \mathcal{C} . Then F induces an equivalence of categories $LF: \text{Ho } A\text{-alg} \rightarrow \text{Ho } FA\text{-alg}$.*

Proof. Let U denote the right adjoint of F . It is clear that F defines a functor from A -algebras to FA -algebras. Indeed, F defines a monoidal functor from A -modules to FA -modules. To see that U defines a functor going the other way, note that we have a natural map $UX \wedge_A UY \rightarrow U(X \wedge_{FA} Y)$ adjoint to the composite

$$F(UX \wedge_A UY) \cong FUX \wedge_{FA} FUY \xrightarrow{\varepsilon \wedge \varepsilon} X \wedge_{FA} Y$$

Thus, if X is an A -algebra, so is UX ; the unit of UX is adjoint to the unit of X using the isomorphism. One can easily check that $F: A\text{-alg} \rightarrow FA\text{-alg}$ is left adjoint to $U: FA\text{-alg} \rightarrow A\text{-alg}$.

The functor U preserves fibrations and trivial fibrations of monoids. It follows that F preserves cofibrations and those trivial cofibrations with source cofibrant

in $A\text{-mod}$. We can then define $(LF)X = F(QX)$ as usual. We have to define $(RU)X = U(RQX)$ however, since RX does not make sense in general. The usual argument shows that $LF: \text{Ho } A\text{-alg} \rightarrow \text{Ho } FA\text{-alg}$ is left adjoint to RU .

The functor U reflects weak equivalences between fibrant objects. A similar argument as in the proof of [5, Corollary 1.3.16] implies that we need only check that the map $X \rightarrow URFX$ is a weak equivalence for cofibrant A -algebras X . Note that FX is still cofibrant, so we do not need to consider $URQFX$, though of course we could do so without difficulty. Here R is a fibrant replacement functor in $FA\text{-alg}$. If we let L denote a fibrant replacement functor in $FA\text{-mod}$, then we know from Theorem 1.8 that $X \rightarrow ULFX$ is a weak equivalence, since X is also cofibrant as an A -module. A simple lifting argument shows that there is a weak equivalence $ULFX \rightarrow RFX$ in $FA\text{-mod}$. Since U preserves weak equivalences between fibrant objects, this gives a weak equivalence $ULFX \rightarrow URFX$. Hence $X \rightarrow URFX$ is a weak equivalence as well. \square

Example 2.7. For example, we can take \mathcal{C} to be the symmetric monoidal model category of simplicial symmetric spectra [7]. Here the monoid axiom holds and everything is small relative to the whole category, so we get a model category of S -algebras. We take \mathcal{D} to be the symmetric monoidal model category of topological symmetric spectra. We do not know whether the monoid axiom holds here. Nonetheless, the geometric realization is a monoidal Quillen equivalence $\mathcal{C} \rightarrow \mathcal{D}$, so defines an equivalence of categories $\text{Ho } S\text{-alg} \rightarrow \text{Ho } S\text{-alg}$ between the homotopy categories of S -algebras.

Note that the homotopy category of an arbitrary model category has a closed action by the homotopy category of simplicial sets [5, Chapter 5]. This result will still hold for $\text{Ho } R\text{-alg}$; one can use the same approach as in [5, Chapter 5], replacing R by RQ everywhere. In particular, there are mapping spaces of monoids. The equivalence of Theorem 2.6 will preserve these mapping spaces.

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