

CHROMATIC PHENOMENA IN THE ALGEBRA OF BP_*BP -COMODULES

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ABSTRACT. We describe the author's research with Neil Strickland on the global algebra and global homological algebra of the category of BP_*BP -comodules. We show, following [HS03a], that the category of $E(n)_*E(n)$ -comodules is a localization, in the abelian sense, of the category of BP_*BP -comodules. This gives analogues of the usual structure theorems, such as the Landweber filtration theorem, for $E(n)_*E(n)$ -comodules. We recall the work of [Hov02a], where an improved version $\text{Stable}(\Gamma)$ of the derived category of comodules over a well-behaved Hopf algebroid (A, Γ) is constructed. The main new result of the paper is that $\text{Stable}(E(n)_*E(n))$ is a Bousfield localization of $\text{Stable}(BP_*BP)$, in analogy to the abelian case.

INTRODUCTION

The object of this paper is to describe some of the author's recent work, much of it joint with Neil Strickland, on comodules over BP_*BP and related Hopf algebroids. The basic idea of this work is to realize the chromatic approach to stable homotopy theory in the algebraic world of comodules. This means, in particular, constructing and understanding the localization functor L_n , or the associated finite localization functor L_n^f , in the abelian category BP_*BP -comod of graded BP_*BP -comodules. It also means constructing L_n and L_n^f in some kind of associated derived category $\text{Stable}(BP_*BP)$ of chain complexes of BP_*BP -comodules. Ultimately, we would like to understand $L_{K(n)}$ in the algebraic setting as well, and this should involve the Morava stabilizer groups. In the present paper, we confine ourselves to L_n and L_n^f .

Topologically, L_n^f is localization away from a finite spectrum of type $n + 1$, and L_n is localization with respect to the homology theory $E(n)$. These functors are probably different in the ordinary stable homotopy category (because Ravenel's telescope conjecture [Rav84] is widely expected to be false), but they turn out to agree on the abelian category of BP_*BP -comodules. We then have the following theorem.

Theorem A. *Let L_n denote localization away from BP_*/I_{n+1} in the category of BP_*BP -comodules. Then there is an equivalence of categories between $E(n)_*E(n)$ -comodules and L_n -local BP_*BP -comodules.*

Note that this is localization in an abelian sense. The localization L_n turns out to be the localization functor that inverts all maps whose kernel and cokernel are v_n -torsion.

This theorem is really a special case of a more general theorem proved in [HS03a]. In general, if (A, Γ) is a flat Hopf algebroid and B is a Landweber exact A -algebra,

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there is an induced flat Hopf algebroid (B, Γ_B) . We prove in [HS03a] that, in this situation, the category of Γ_B -comodules is always equivalent to some localization of the category of Γ -comodules. In the case of BP_*BP , we give a partial classification of such localizations. In particular, $E(n)_*E(n)$ can be replaced by E_*E for any Landweber exact commutative ring spectrum E with $E_*/I_n \neq 0$ and $E_*/I_{n+1} = 0$. This tells us that all such theories E have equivalent categories of E_*E -comodules, even though the categories of E_* -modules can be drastically different. It also leads to structural results about E_*E -comodules analogous to those of Landweber [Lan76] for BP_*BP -comodules.

To extend this result to the derived category setting, we first must decide what we mean by the derived category. The ordinary derived category, obtained by inverting homology isomorphisms, is usually badly behaved. For example, the analogue of S^0 in the derived category of BP_*BP -comodules is BP_* thought of as a complex concentrated in degree 0, but this is not a small object (see the introduction to Section 3).

The following is a corollary of the main result of [Hov02a].

Theorem B. *Suppose E is a commutative ring spectrum that is Landweber exact over BP . Then there is a bigraded monogenic stable homotopy category $\text{Stable}(E_*E)$ such that*

$$\pi_{**}S^0 \cong \text{Ext}_{E_*E}^{**}(E_*, E_*).$$

In particular, the sphere, which is E_* concentrated in degree 0, is a small object of $\text{Stable}(E_*E)$. This stable homotopy category is the homotopy category of a model structure on chain complexes of E_*E -comodules. The weak equivalences are homotopy isomorphisms, where homotopy is suitably defined. Every cofibrant object is dimensionwise projective over E_* and every complex of relatively injective comodules is fibrant. Just as in the ordinary stable homotopy category, there exist interesting nontrivial complexes with no homology.

On $\text{Stable}(BP_*BP)$, we define L_n^f to be finite localization away from BP_*/I_{n+1} , which turns out to be a small object in $\text{Stable}(BP_*BP)$. We define L_n to be Bousfield localization with respect to the homology theory corresponding to $E(n)_*$. In $\text{Stable}(BP_*BP)$, the homology functor H is represented by BP_*BP , so is somewhat analogous to the BP -homology of a spectrum. Then the homology theory corresponding to $E(n)_*$ is in fact $HE(n)$, ordinary homology with coefficients in $E(n)_*$. Because of this, some of the things one would expect to be true about L_n are false. For example, the two localizations L_n^f and L_n on $\text{Stable}(BP_*BP)$ are definitely different in general, so the most naive version of the telescope conjecture is false in $\text{Stable}(BP_*BP)$. Also, L_n is not a smashing localization, though L_n^f is. Thus L_n is less important in $\text{Stable}(BP_*BP)$ than L_n^f .

These statements are also true in the category $\text{Stable}(E(n)_*E(n))$, where L_n^f is the identity functor (as it is localization away from $E(n)_*/I_{n+1} = 0$), and L_n is localization with respect to ordinary homology (which already has coefficients in $E(n)_*$). Thus $L_n \text{Stable}(E(n)_*E(n))$ is the classical unbounded derived category of $E(n)_*E(n)$ -comodules.

The main new result of this paper is then the following theorem.

Theorem C. *There is an equivalence of stable homotopy categories between the localization $L_n^f \text{Stable}(BP_*BP)$ and $\text{Stable}(E(n)_*E(n))$.*

As above in Theorem A, we can replace $E(n)$ in Theorem C by any Landweber exact commutative ring spectrum E with $E_*/I_n \neq 0$ and $E_*/I_{n+1} = 0$. Theorem C also implies a similar equivalence between $L_n\text{Stable}(BP_*BP)$ and $L_n\text{Stable}(E(n)_*E(n))$.

From a computational point of view, Theorem C gives rise to the following change of rings theorem.

Theorem D. *Suppose that M is a finitely presented BP_*BP -comodule and N is a BP_*BP -comodule such that $N = L_n N$ and the right derived functors $L_n^i N$ are 0 for $i > 0$. Then there is a change of rings isomorphism*

$$\text{Ext}_{BP_*BP}^{**}(M, N) \cong \text{Ext}_{E(n)_*E(n)}(E(n)_* \otimes_{BP_*} M, E(n)_* \otimes_{BP_*} N).$$

This change of rings theorem includes the Miller-Ravenel change of rings theorem [MR77] and the change of rings theorem of the author and Sadofsky [HS99] as special cases. Also, $E(n)$ can be replaced by any Landweber exact commutative ring spectrum E with $E_*/I_n \neq 0$ and $E_*/I_{n+1} = 0$.

Because the basic structure of the abelian category of comodules over a flat Hopf algebroid is not as well known as it should be, we first summarize this in Section 1. The results in this section were mostly proved in [Hov02a]. We then describe our proof of Theorem A and related results about $E(n)_*E(n)$ -comodules in Section 2. Further details can be found in [HS03a]. We introduce the stable homotopy category of comodules in Section 3, describing the proof of Theorem B and looking at some particular features of $\text{Stable}(BP_*BP)$ and $\text{Stable}(E(n)_*E(n))$. Some of the material in this section can be found in [Hov02a], but some of it is new. We discuss the relation between $\text{Stable}(BP_*BP)$ and $\text{Stable}(E(n)_*E(n))$ in Section 4, where we prove Theorems C and D. All of the results in this section are new.

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1. COMODULES

The object of this section is to give an overview of the structural properties of the category Γ -comod. We begin by recalling the structure maps of a Hopf algebroid (A, Γ) .

A Hopf algebroid (A, Γ) has the following structure maps, which are all maps of commutative rings.

- The *counit* $\epsilon: \Gamma \rightarrow A$, corepresenting the identity map of an object.
- The *left unit* $\eta_L: A \rightarrow \Gamma$, corepresenting the source of a morphism.
- The *right unit* $\eta_R: A \rightarrow \Gamma$, corepresenting the target of a morphism.
- The *diagonal* $\Delta: \Gamma \rightarrow \Gamma \otimes_A \Gamma$, corepresenting the composite of two composable morphisms. Note that this is a tensor product of A -bimodules, with the left A -module structure given by η_L and the right A -module structure given by η_R .
- The *conjugation* $\chi: \Gamma \rightarrow \Gamma$, corepresenting the inverse of a morphism.

There are many relations between these structure maps, but they are all easily obtained from corresponding facts about groupoids. For example, the source and target of the identity morphism at x are both x , so $\epsilon\eta_L = \epsilon\eta_R = 1$.

Note that the conjugation is best thought of as a map $\chi: \Gamma \rightarrow \tilde{\Gamma}$, where $\tilde{\Gamma}$ denotes Γ with the opposite A -bimodule structure, so that A acts on the left on $\tilde{\Gamma}$ by η_R . Similarly, the multiplication map is best thought of as $\mu: \Gamma \otimes_A \tilde{\Gamma} \rightarrow \Gamma$, although we could also think of it as having domain $\tilde{\Gamma} \otimes_A \Gamma$. With this convention, the fact that composition of a map with its inverse is the identity gives the following commutative diagram,

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\Delta} & \Gamma \otimes_A \Gamma & \xrightarrow{1 \otimes \chi} & \Gamma \otimes_A \tilde{\Gamma} \\ \parallel & & & & \downarrow \mu \\ \Gamma & \xrightarrow{\epsilon} & A & \xrightarrow{\eta_L} & \Gamma \end{array}$$

and a similar diagram involving $\chi \otimes 1$ and η_R . This is a great deal simpler than the corresponding diagram in [Rav86, Definition A1.1.1(f)].

We recall that a Γ -comodule is a left A -module M together with a counital and coassociative coaction map $\psi: M \rightarrow \Gamma \otimes_A M$ of left A -modules, where again Γ is an A -bimodule. A map of comodules is a map of A -modules that preserves the coaction, so we get a category Γ -comod of Γ -comodules.

We then have the following proposition [Rav86, Proposition 2.2.8], which explains the importance of Hopf algebroids and comodules in algebraic topology.

Proposition 1.1. *Suppose E is a ring spectrum such that E_*E is a commutative ring that is flat over E_* . Then (E_*, E_*E) is a Hopf algebroid, and E_*X is naturally an E_*E -comodule for X a spectrum.*

Other good examples of Hopf algebroids include Hopf algebras, which are just Hopf algebroids where $\eta_L = \eta_R$. In particular, a commutative ring A can be thought of as the *discrete* Hopf algebroid (A, A) ; a comodule over a discrete Hopf algebroid is just an A -module. Also, if G acts on a commutative ring R by ring automorphisms, the ring of R -valued functions on G is a Hopf algebroid. The right unit is defined by $\eta_R(r)(g) = g(r)$. This is dual to the twisted group ring $R[G]$.

We will summarize the properties of Γ -comod in the following theorem, but for it to make sense we need to recall some definitions.

- Definition 1.2.**
- (a) A Hopf algebroid (A, Γ) is **flat** if η_R makes Γ into a flat A -module. The conjugation shows that it is equivalent to assume η_L is flat. Since ϵ is a right inverse for both η_L and η_R , both of these maps are then faithfully flat.
 - (b) A category is **complete** if it has all small limits, and **cocomplete** if it has all small colimits. It is **bicomplete** if it is both complete and cocomplete.
 - (c) Given a regular cardinal λ , a category \mathcal{I} is said to be **λ -filtered** if every subcategory \mathcal{J} of \mathcal{I} with fewer than λ morphisms has an upper bound in \mathcal{I} ; that is, there is an object C in \mathcal{I} and a natural transformation from the inclusion functor $\mathcal{J} \rightarrow \mathcal{I}$ to the constant \mathcal{J} -diagram at C . An object M of a cocomplete category \mathcal{C} is said to be **λ -presented** if $\mathcal{C}(M, -)$ commutes with λ -filtered colimits.
 - (d) Suppose \mathcal{C} is a closed symmetric monoidal category with monoidal structure $X \wedge Y$, unit A , and closed structure $F(X, Y)$. An object M in \mathcal{C} is said to be

dualizable if the natural map $F(M, A) \wedge X \rightarrow F(M, X)$ is an isomorphism for all $X \in \mathcal{C}$. An object M is said to be **invertible** if there is an N and an isomorphism $M \wedge N \cong A$.

Theorem 1.3. *Suppose (A, Γ) is a flat Hopf algebroid. Then Γ -comod is a bicomplete, closed symmetric monoidal abelian category. We also have:*

- (a) *Filtered colimits are exact.*
- (b) *Given a regular cardinal λ and a comodule M , M is λ -presented if and only if M is λ -presented as an A -module.*
- (c) *For any comodule M , there is a cardinal λ such that M is λ -presented.*
- (d) *A comodule M is dualizable if and only if it is projective and finitely generated over A .*
- (e) *A comodule M is invertible under the symmetric monoidal product if and only if it is invertible as an A -module.*

We denote the symmetric monoidal structure by $M \wedge N$ with unit A and the closed structure by $F(M, N)$.

This theorem is a summary of the results of [Hov02a, Section 1]. We will just discuss some of the issues that arise. First of all, left adjoints are generally easy to construct, since the forgetful functor from Γ -comod to A -mod is itself a left adjoint (its right adjoint is the extended comodule functor discussed in the following paragraph). Thus, one generally forms the left adjoint in A -mod and notices that it has a natural comodule structure. This is true for colimits and for the symmetric monoidal structure $M \wedge N$. This is defined to be $M \otimes_A N$, the tensor product of **left** A -modules, with the coaction given as the composite

$$M \otimes_A N \xrightarrow{\psi \otimes \psi} (\Gamma \otimes_A M) \otimes_A (\Gamma \otimes_A N) \xrightarrow{g} \Gamma \otimes_A M \otimes_A N$$

where $g(x \otimes m \otimes y \otimes n) = xy \otimes m \otimes n$.

The key to constructing right adjoints is the extended comodule functor from A -mod to Γ -comod that takes M to $\Gamma \otimes_A M$, with coaction $\Delta \otimes 1$. This is the right adjoint to the forgetful functor. As such, it is generally easy to define a desired right adjoint R on extended comodules. For example, one can easily see that we must define the product of extended comodules by

$$\prod_{\Gamma} (\Gamma \otimes_A M_i) \cong \Gamma \otimes_A \prod M_i,$$

and the closed structure with target an extended comodule by

$$F(M, \Gamma \otimes_A N) \cong \Gamma \otimes_A \text{Hom}_A(M, N).$$

It is less easy to see how one defines these right adjoints on maps between extended comodules that are not necessarily extended maps, but this can generally be done. Having done this, we use the exact sequence of comodules

$$0 \rightarrow M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\psi g} \Gamma \otimes_A N,$$

where $g: \Gamma \otimes_A M \rightarrow N$ is the cokernel of ψ , to define $RM = \ker R(\psi g)$. Since R is supposed to be a right adjoint, it must be left exact, so we must define R in this way.

Note that if M is an A -module and N is a Γ -comodule, we have the two tensor products $\Gamma \otimes_A (M \otimes_A N)$ and $(\Gamma \otimes_A M) \wedge N$. It is useful to know that these are the same [Hov02a].

Lemma 1.4. *Suppose (A, Γ) is a flat Hopf algebroid, M is an A -module and N is a Γ -comodule. Then there is a natural isomorphism of comodules*

$$(\Gamma \otimes_A M) \wedge N \rightarrow \Gamma \otimes_A (M \otimes_A N).$$

Although Theorem 1.3 indicates that the category of Γ -comodules is a very well-behaved abelian category, one obvious property is missing, and that is the existence of a set of generators. Recall that a set of objects \mathcal{G} is said to **generate** an abelian category \mathcal{C} if, whenever f is a nonzero map in \mathcal{C} , there exists an object $G \in \mathcal{G}$ such that $\mathcal{C}(G, f)$ is also nonzero. For example, A is a generator of $A\text{-mod}$. This issue of generators is already complicated for Hopf algebras; for a finite group G and a field k , the natural generators for the category of $k[G]$ -modules (which is isomorphic to the category of comodules over the ring of k -valued functions on G) are the simple $k[G]$ -modules. There is no canonical description of these in general. However, any simple $k[G]$ -module is finitely generated, and of course projective, over k . Referring to part (d) of Theorem 1.3, one might then expect that the set of isomorphism classes of dualizable comodules forms a set of generators for $\Gamma\text{-comod}$. Sadly, this appears to be false in general, so we need a hypothesis.

Definition 1.5. A Hopf algebroid (A, Γ) is called an **Adams Hopf algebroid** when Γ is a filtered colimit of dualizable comodules.

This hypothesis is really due to Adams [Ada74, Section III.13], who used it for the Hopf algebroid (E_*, E_*E) to set up universal coefficient spectral sequences. We learned it from [GH00], as well as the following lemma.

Lemma 1.6. *Suppose (A, Γ) is an Adams Hopf algebroid. Then it is flat, and the dualizable comodules generate the category of Γ -comodules.*

In categorical language, the category of comodules over an Adams Hopf algebroid is a locally finitely presentable Grothendieck category.

All of the Hopf algebroids that commonly arise in algebraic topology, as well as all Hopf algebras over fields, are known to be Adams [Hov02a, Section 1.4].

Note that it may be that one can take smaller sets of generators than all of the dualizable comodules. For example, the set $\{BP_*X_n\}$ will serve as a set of generators for BP_*BP -comodules, where X_n is the $2n$ -skeleton of BP . However, BP_* by itself is definitely not a generator for the category of BP_*BP -comodules. To see this, let PM be the set of primitives in a BP_*BP -comodule M . Then one can easily check that if BP_* is a generator of the category of BP_*BP -comodules then any comodule map that is surjective after applying P is in fact surjective. In particular, the map

$$\bigoplus_{x \in PM} \Sigma^{|x|} BP_* \xrightarrow{f} M$$

would be surjective. This is easily seen to be false for $M = BP_*(\mathbb{C}P^2)$, for example.

Remark 1.7. Theorem 1.3 lists the good points of the category $\Gamma\text{-comod}$; we now list some of the bad points of $\Gamma\text{-comod}$.

- (a) The forgetful functor from $\Gamma\text{-comod}$ to A -modules, or even down to abelian groups, does NOT have a left adjoint; there is no free comodule functor.
- (b) $\Gamma\text{-comod}$ does not, in general, have enough projectives. If we take $(A, \Gamma) = (\mathbb{F}_p, \mathcal{A})$, where \mathcal{A} denotes the dual Steenrod algebra, it is generally believed that there are no nonzero projective comodules.

- (c) If (A, Γ) is not a Hopf algebra, that is if η_L and η_R are not equal, then the forgetful functor from Γ -comod to A -mod is not in general surjective on objects. For example, there is no BP_*BP -comodule structure on $v_n^{-1}BP_*$ for $n > 0$ [JY80, Proposition 2.9].
- (d) Products are not in general exact. Hence the inverse limit functor \lim_{Γ} on sequences

$$\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$$

may have nonzero derived functors \lim_{Γ}^i for all $i > 0$.

Because there are not enough projective comodules, the homological algebra of comodules always involves injectives, or, better, relative injectives. Because the forgetful functor is exact, if I is an injective A -module, then $\Gamma \otimes_A I$ is an injective Γ -comodule. From this it is easy to check that there are enough injectives. However, injective A -modules are complicated, whereas relative injectives have much better properties.

A comodule I is defined to be **relatively injective** if Γ -comod $(-, I)$ takes A -split short exact sequences of comodules to short exact sequences. The following proposition sums up the properties of relative injectives and is well-known; details can be found in [Hov02a, Section 3.1].

Proposition 1.8. *Suppose (A, Γ) is a flat Hopf algebroid.*

- (a) *The relatively injective Γ -comodules are the retracts of extended comodules.*
- (b) *The coaction $\psi: M \rightarrow \Gamma \otimes_A M$ defines an A -split embedding of M into a relatively injective comodule.*
- (c) *Relatively injective comodules are closed under coproducts and products.*
- (d) *If I is relatively injective, so is $I \wedge M$ and $F(M, I)$ for all comodules I .*
- (e) *If P is a comodule that is projective over A , and I is relatively injective, then $\text{Ext}_{\Gamma}^n(P, I) = 0$ for all $n > 0$.*

We take this proposition to mean that, to understand $\text{Ext}_{\Gamma}^*(M, N)$, we must simultaneously resolve M by comodules that are projective over A and N by relative injectives. We return to this point in Section 3.

We close this section with a brief description of naturality. There is, of course, a natural notion of a map $\Phi: (A, \Gamma) \rightarrow (B, \Sigma)$ of Hopf algebroids. The map Φ corepresents a natural functor of groupoids, so consists of ring maps $\Phi_0: A \rightarrow B$ and $\Phi_1: \Gamma \rightarrow \Sigma$ satisfying certain conditions. Such a map induces a symmetric monoidal functor $\Phi_*: \Gamma$ -comod \rightarrow Σ -comod that takes M to $B \otimes_A M$, with comodule structure given by the composite

$$B \otimes_A M \xrightarrow{1 \otimes \psi} B \otimes_A \Gamma \otimes_A M \xrightarrow{g \otimes 1} \Sigma \otimes_A M \cong \Sigma \otimes_B (B \otimes_A M)$$

where $g(b \otimes x) = b\Phi_1(x)$. It is clear that Φ_* preserves colimits, so should have a right adjoint Φ^* . It does, but, as usual, Φ^* is hard to define. We define $\Phi^*(\Sigma \otimes_B M) = \Gamma \otimes_A M$, and then extend this definition to all Σ -comodules in the same way we did for the product of comodules.

An important new feature that arises in the study of Hopf algebroids is the notion of weak equivalence.

Definition 1.9. A map Φ of Hopf algebroids is defined to be a **weak equivalence** if Φ_* is an equivalence of categories.

If Φ is a weak equivalence between discrete Hopf algebroids, then Φ is an isomorphism, but a central point of the author's work on Hopf algebroids is that there are important non-trivial weak equivalences of Hopf algebroids that are not discrete.

In general, we have the following characterization of weak equivalences.

Theorem 1.10. *The map Φ of Hopf algebroids is a weak equivalence if and only if the composite*

$$A \xrightarrow{\eta_R} \Gamma \xrightarrow{\Phi_0 \otimes 1} B \otimes_A \Gamma$$

is a faithfully flat ring extension and the map

$$B \otimes_A \Gamma \otimes_A B \rightarrow \Sigma$$

that takes $b \otimes x \otimes b'$ to $\eta_L(b)\Phi_1(x)\eta_R(b')$ is a ring isomorphism.

The “if” half of this theorem is the main result of [Hov02b]. The “only if” half is much easier and was proven in [HS03a].

This theorem has a better formulation. Hollander [Hol01] has constructed a model structure on presheaves of groupoids on a Grothendieck topology \mathcal{C} ; the fibrant objects are stacks; see also [Jar01]. In particular, we can take our Grothendieck site to be the flat topology on \mathbf{Aff} , the opposite category of commutative rings (with a cardinality bound so we get a small category). In this topology, a cover of R is a finite collection of flat extensions S_i of R such that $\coprod S_i$ is faithfully flat over R . Any Hopf algebroid (A, Γ) defines a presheaf of groupoids $\mathrm{Spec}(A, \Gamma)$ by definition; this presheaf is in fact a sheaf in the flat topology by faithfully flat descent [Hov02b]. We can then rephrase Theorem 1.10 as follows.

Corollary 1.11. *A map Φ of Hopf algebroids is a weak equivalence if and only if $\mathrm{Spec} \Phi$ is a weak equivalence of sheaves of groupoids in the flat topology.*

This point of view suggests that one should reconsider the results of this section for quasi-coherent sheaves over a sheaf of groupoids, since a quasi-coherent sheaf over $\mathrm{Spec}(A, \Gamma)$ is the same thing as a Γ -comodule [Hov02b]. We have not carried out this program. One reason for this is that we don't see any clear applications. But another reason is that we do not know whether the Adams condition is invariant under weak equivalence. The difficulty is that, while dualizable comodules and filtered colimits are preserved by weak equivalences, Γ is not. One could simply demand that dualizable sheaves generate the category of quasi-coherent sheaves, but again we do not know if this is sufficient to provide a useful theory, or even whether it holds for interesting non-affine sheaves of groupoids.

It would be interesting to know if there are equivalences of categories of comodules that are not given by maps of Hopf algebroids, as occurs in Morita theory. Since Hopf algebroids are a generalization of **commutative** rings, and there are no non-trivial Morita equivalences of commutative rings, it is reasonable to guess that every equivalence of categories of comodules is a zig-zag of weak equivalences.

2. LANDWEBER EXACT ALGEBRAS

The object of this section is to study the relation between Γ -comodules and Γ_B -comodules, where B is a Landweber exact A -algebra. The main application is to the relation between BP_*BP -comodules and $E(n)_*E(n)$ -comodules. In particular, we sketch the proof of Theorem A and its corollaries in this section. More details can be found in [HS03a].

Given a Hopf algebroid (A, Γ) , an A -algebra B is said to be **Landweber exact** over A if $B \otimes_A (-)$ takes exact sequences of Γ -comodules to exact sequences of B -modules. This is called Landweber exactness because Landweber gave a characterization of Landweber exact BP_* -algebras in his famous Landweber exact functor theorem [Lan76]. One can check that B is Landweber exact over A if and only if the composite

$$A \xrightarrow{\eta_R} \Gamma \rightarrow B \otimes_A \Gamma$$

is a flat ring extension. This condition is reminiscent of the characterization of weak equivalences given in Theorem 1.10. We can make it even more so by defining

$$\Gamma_B = B \otimes_A \Gamma \otimes_A B.$$

We then have the following lemma, which is easy to prove but can also be found in [HS03a].

Lemma 2.1. *Suppose (A, Γ) is a Hopf algebroid and B is an A -algebra. Then (B, Γ_B) is a Hopf algebroid, and the evident map $(A, \Gamma) \rightarrow (B, \Gamma_B)$ is a map of Hopf algebroids. If (A, Γ) is flat and B is Landweber exact, then (B, Γ_B) is a flat Hopf algebroid.*

Thus, if B is Landweber exact over A , the map

$$\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$$

is almost a weak equivalence, in that

$$A \xrightarrow{\eta_R} \Gamma \rightarrow B \otimes_A \Gamma$$

is flat, and

$$B \otimes_A \Gamma \otimes_A B \rightarrow \Gamma_B$$

is an isomorphism. The only thing stopping Φ from being a weak equivalence is that $B \otimes_A (-)$ may not be faithful on the category of Γ -comodules. The idea of the following theorem, proved in [HS03a], is that we can force Φ to be faithful by localizing the category of Γ -comodules.

Theorem 2.2. *Suppose (A, Γ) is a flat Hopf algebroid, and B is Landweber exact over A . Then the map*

$$\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$$

yields an equivalence

$$\Phi_*: L_{\mathcal{T}}(\Gamma\text{-comod}) \rightarrow \Gamma_B\text{-comod}$$

where $L_{\mathcal{T}}(\Gamma\text{-comod})$ is the localization of Γ -comod with respect to the hereditary torsion theory \mathcal{T} consisting of all Γ -comodules M such that $B \otimes_A M = 0$.

A **hereditary torsion theory** is just a full subcategory closed under subobjects, quotient objects, extensions, and arbitrary direct sums. The localization $L_{\mathcal{T}}$ is obtained by inverting all maps f of Γ -comodules whose kernel and cokernel are in \mathcal{T} . Note that the Hopf algebroids that arise in algebraic topology are graded, so B will be a graded A -algebra, and our hereditary torsion theories will also be graded, in the sense that M is in \mathcal{T} if and only if all shifts of M are in \mathcal{T} .

Because it is so surprisingly easy, we will give the proof of Theorem 2.2.

Proof. Consider the natural transformation

$$\epsilon_M: \Phi_* \Phi^* M \rightarrow M.$$

We claim that this map is a natural isomorphism. One can check this by calculation for extended Σ -comodules M . Since ϵ is a natural transformation of left exact functors (because B is Landweber exact), and every Σ -comodule is the kernel of a map of extended comodules, ϵ_M is an isomorphism for all M .

After this, the rest of the proof of Theorem 2.2 is purely formal. A priori, the category $L_{\mathcal{T}}(\Gamma\text{-comod})$ may not be an actual category, since it may not have small Hom sets, always a danger with localization. However, it does exist in a higher universe. The natural transformation

$$\eta_M: M \rightarrow \Phi^* \Phi_* M$$

becomes an isomorphism upon applying Φ_* , and therefore, since Φ_* is exact, the kernel and cokernel of η_M are in \mathcal{T} . This gives us the desired equivalence, and incidentally shows that $L_{\mathcal{T}}(\Gamma\text{-comod})$ actually does exist as an honest category, since it is equivalent to $\Gamma_B\text{-comod}$. \square

Theorem 2.2 gives the following corollary, which it is difficult to imagine proving directly.

Corollary 2.3. *Suppose (A, Γ) is a flat Hopf algebroid, B is Landweber exact over A , and every nonzero Γ -comodule has a primitive. Then every nonzero Γ_B -comodule has a primitive. In particular, every $E(n)_* E(n)$ -comodule has a primitive.*

This corollary is immediate, as $L_{\mathcal{T}}(\Gamma\text{-comod})$ is the full subcategory of Γ -comod consisting of the local objects.

To get further information, we need to identify the hereditary torsion theories that can arise. Let \mathcal{T}_n denote the collection of all v_n -torsion $BP_* BP$ -comodules, so that \mathcal{T}_0 is the collection of all p -torsion comodules, and \mathcal{T}_{-1} is the collection of all comodules. One can easily check that \mathcal{T}_n is a hereditary torsion theory. It is less obvious, but true, that \mathcal{T}_n is the smallest graded hereditary torsion theory containing BP_*/I_{n+1} .

Theorem 2.4. *Let \mathcal{T} be a graded hereditary torsion theory of $BP_* BP$ -comodules. If \mathcal{T} contains a nonzero finitely presented comodule, then $\mathcal{T} = \mathcal{T}_n$ for some n .*

This theorem is proved in [HS03a], using the ideas behind the Landweber filtration theorem. Note that this theorem explains why L_n and L_n^f agree on the category of $BP_* BP$ -comodules. The only reasonable definition of L_n is localization with respect to the hereditary torsion theory of all comodules M such that $E(n)_* \otimes_{BP_*} M = 0$, and L_n^f is localization with respect to the hereditary torsion theory generated by BP_*/I_{n+1} . Both of these torsion theories are \mathcal{T}_n .

Because of this theorem, it is natural to make the following definition.

Definition 2.5. Suppose B is a BP_* -algebra. Define the **height** of B to be the largest integer n such that B/I_n is nonzero. If there is no such n , define the height of B to be infinite.

From a formal group law point of view, the height of B is the largest possible height of any specialization of the formal group law of B . So the height of $E(n)$ is n , and the height of BP itself is ∞ .

Here is the main theorem of [HS03a].

Theorem 2.6. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B is a graded Landweber exact A -algebra of height $n \leq \infty$. Then the functor $M \mapsto B \otimes_A M$ defines an equivalence of categories*

$$L_n(\Gamma\text{-comod}) \rightarrow \Gamma_B\text{-comod},$$

where L_n is localization with respect to \mathcal{T}_n for $n < \infty$ and L_∞ is the identity localization.

This theorem is almost a corollary of Theorem 2.2 and Theorem 2.4, except for the infinite height case. We do not have a classification of graded hereditary torsion theories of BP_*BP -comodules that do not contain a nonzero finitely presented comodule. There are probably uncountably many such torsion theories. However, if B is Landweber exact and $B \otimes_A M = 0$ for some nonzero M , then $B/I_\infty B = 0$. Indeed, M must have a nonzero primitive, and so we conclude by Landweber exactness that $B/IB = 0$ for some proper invariant ideal I . Since $I \subseteq I_\infty$, it follows that $B/I_\infty B = 0$. But this means that $1 \in I_\infty B$, so $1 \in I_n B$ for some n . Thus $B \otimes_A A/I_n = 0$. This proves that if B has infinite height, then $B \otimes_A (-)$ does not kill any nonzero BP_*BP -comodules.

The following corollary is immediate.

Corollary 2.7. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B and B' are both Landweber exact A -algebras of the same height. Then the category of Γ_B -comodules is equivalent to the category of $\Gamma_{B'}$ -comodules.*

In particular, the category of $E(n)_*E(n)$ -comodules is equivalent to the category of $v_n^{-1}BP_*(v_n^{-1}BP)$ -comodules, even though the category of $E(n)_*$ -modules is very different from the category of $v_n^{-1}BP_*$ -modules. One way to think of the Miller-Ravenel change of rings theorem [MR77, Theorem 3.10] as an isomorphism of certain Ext groups in these two categories. This is now obvious; the Ext groups are isomorphic because the categories they are taken in are equivalent.

We point out that the equivalence of categories of comodules in Corollary 2.7 is in fact induced by a zig-zag of weak equivalences of Hopf algebroids. Any map $B \rightarrow B'$ of Landweber exact BP_* -algebras of the same height induces a weak equivalence of Hopf algebroids

$$(B, \Gamma_B) \rightarrow (B', \Gamma_{B'}),$$

where we are still denoting BP_*BP by Γ . If B and B' are Landweber exact BP_* -algebras of the same height, there may not be a map of BP_* -algebras between them. However, if we let $C = B \otimes_{BP_*} \Gamma \otimes_{BP_*} B'$, then C has a left and right BP_* -algebra structure, which we denote by C_L and C_R . There are maps of BP_* -algebras $B \rightarrow C_L$ and $B' \rightarrow C_R$, and conjugation induces an isomorphism $C_L \rightarrow C_R$. Since C_L is also Landweber exact, of the same height as B and B' , this yields the desired zig-zag of weak equivalences.

To further understand the structure of the category of $E(n)_*E(n)$ -comodules, we would like to understand the localization functor L_n better. The following theorem is a summary of the results of [HS03b], and is joint work of the author and Strickland.

Theorem 2.8. (a) *A comodule M is L_n -local if and only if*

$$\mathrm{Hom}_{BP_*BP}^*(BP_*/I_{n+1}, M) = \mathrm{Ext}_{BP_*BP}^{1,*}(BP_*/I_{n+1}, M) = 0,$$

which is true if and only if

$$\mathrm{Hom}_{BP_*}^*(BP_*/I_{n+1}, M) = \mathrm{Ext}_{BP_*}^{1,*}(BP_*/I_{n+1}, M) = 0,$$

- (b) L_n , thought of as an endofunctor of the category of BP_*BP -comodules, is left exact and preserves finite limits, filtered colimits, and arbitrary direct sums. It has right derived functors which we denote L_n^i .
- (c) For $i > 0$, $L_n^i(M)$ is isomorphic to the $i+1$ st local cohomology group of the BP_* -module M with respect to I_{n+1} . In particular, $L_n^i(M) = 0$ for $i > n$.
- (d) Suppose $m < n$ and M is a v_{m-1} -torsion comodule on which (v_m, v_{m+1}) is a regular sequence. Then M is L_n -local.
- (e) If v_m acts invertibly on a comodule M for some $m \leq n$, then M is L_n -local and $L_n^i M = 0$ for all $i > 0$.
- (f) If a comodule M is v_{n-1} -torsion, then $L_n M = v_n^{-1} M$.
- (g) We have

$$L_n(BP_*/I_k) = \begin{cases} BP_*/I_k & k < n, \\ v_n^{-1} BP_*/I_n & k = n, \\ 0 & k > n. \end{cases}$$

and, for $i, n > 0$,

$$L_n^i(BP_*/I_k) = \begin{cases} BP_*/(p, v_1, \dots, v_{k-1}, v_k^\infty, \dots, v_n^\infty) & i = n - k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

These derived functors L_n^i can be used to compute $BP_*(L_n X)$ from $BP_* X$ by means of a spectral sequence.

Theorem 2.9. *Let X be a spectrum. There is a natural spectral sequence $E_*^{**}(X)$ with $d_r: E_r^{s,t} \rightarrow E_r^{s+t, t+r-1}$ and E_2 -term $E_2^{s,t}(X) \cong (L_n^s BP_* X)_t$, converging to $BP_{t-s}(L_n X)$. This is a spectral sequence of BP_*BP -comodules, in the sense that $E_r^{s,*}$ is a graded BP_*BP -comodule for all $r \geq 2$ and $d_r: E_r^{s,*} \rightarrow E_r^{s+r,*}$ is a BP_*BP -comodule map of degree $r-1$. Furthermore, every element in $E_2^{0,*}$ that comes from $BP_* X$ is a permanent cycle.*

This theorem is proved in [HS03b]. It is very closely related to the local cohomology spectral sequence of Greenlees [Gre93] and Greenlees and May [GM95]. One way of putting it is that we show that the Greenlees spectral sequence is a spectral sequence of comodules in this case. When $X = S^0$, this implies that the spectral sequence collapses with no extensions, and we recover Ravenel's computation [Rav84] of $BP_* L_n S^0$.

We now derive some corollaries of Theorem B, proved in [HS03a].

Corollary 2.10. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B is a Landweber exact A -algebra of height n . Then*

$$\Gamma_B\text{-comod}(B, B/I_m) = \begin{cases} \mathbb{Z}_{(p)} & n > m = 0 \\ \mathbb{Q} & n = m = 0 \\ \mathbb{F}_p[v_m] & n > m > 0 \\ \mathbb{F}_p[v_m, v_m^{-1}] & n = m > 0. \end{cases}$$

Proof. Let $\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$ be the evident map of Hopf algebroids, so that $L_n = \Phi^* \Phi_*$ by Theorem 2.6. Then we have

$$\Gamma_B\text{-comod}(B, B/I_m) \cong \Gamma\text{-comod}(A, \Phi^*(B/I_m)) \cong \Gamma\text{-comod}(A, L_n(A/I_m)),$$

so the result follows from Theorem 2.8 and the analogous calculation for $B = BP_*$ [Rav86, Theorem 4.3.2]. \square

This corollary in turn gives rise to the expected structural results about Γ_B -comodules, proved in [HS03a].

Theorem 2.11. *Let $(A, \Gamma) = (BP_*, BP_*BP)$, and suppose B is a Landweber exact A -algebra of height n .*

- (a) *If I is an invariant radical ideal in B , then $I = I_m$ for some $m \leq n$.*
- (b) *If M is a finitely presented Γ_B -comodule, then there is a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of M by subcomodules such that, for each $s \leq t$, there is a $k \leq n$ and an r such that $M_s/M_{s-1} \cong s^r B/I_k B$.

Proof. Note that the theorem is invariant under the equivalences of categories of Theorem 2.6. Thus we might as well assume that $B = E(n)_*$ or BP_* , and in case $B = BP_*$ the result is due to Landweber [Lan76]. So we assume $B = E(n)_*$. Now, for part (a), suppose I is an invariant radical ideal, and choose the largest k such that $I_k \subseteq I$. If $I_k \neq I$, the comodule I/I_k must have a nonzero primitive y . This primitive must also be a primitive in $E(n)_*/I_k$. By Corollary 2.10, it must be a power of v_k . Since I is radical, this means that $I_{k+1} \subseteq I$. This contradiction implies that $I_k = I$.

For part (b), we construct the filtration M_i by induction on i , taking $M_0 = 0$. Having built M_i , if $M_i \neq M$, we choose a nonzero primitive y in M/M_i . We claim that some multiple z of y is a primitive whose annihilator is I_k for some $k \leq n$. Indeed, if y is p -torsion, then we can multiply y by a power of p to obtain a nonzero primitive y_1 with $py_1 = 0$. Since v_1 is a primitive mod p , if y_1 is v_1 -torsion we can multiply y_1 by power of v_1 to obtain a nonzero primitive y_2 with $py_2 = v_1 y_2 = 0$. Continuing in this fashion, we end up with a primitive z such that $I_k z = 0$ and z is not v_k -torsion. Corollary 2.10 implies that $\text{Ann } z = I_k$, as required.

We now choose an element w in M whose image in M/M_i is z , and let M_{i+1} denote the subcomodule generated by M_i and w . Then $M_{i+1}/M_i \cong s^r E(n)_*/I_k$, where r is the degree of z . Since M is finitely presented over the Noetherian ring $E(n)_*$, this process must stop, and so $M_t = M$ for some t . \square

3. THE STABLE HOMOTOPY CATEGORY $\text{Stable}(\Gamma)$

The object of this section is to discuss the construction and basic properties of the stable homotopy category $\text{Stable}(\Gamma)$ associated to an Adams Hopf algebroid (A, Γ) . In practice, we are most interested in $\text{Stable}(E_*E)$ for E a commutative Landweber exact ring spectrum. This means our Hopf algebroids should be graded, and the stable homotopy category $\text{Stable}(\Gamma)$ should be bigraded. However, the grading just adds unnecessary complexity to the notation, so we will forget about it for most of this section.

Construction of $\text{Stable}(\Gamma)$. The usual derived category $\mathcal{D}(\Gamma)$ of Γ -comodules is obtained from $\text{Ch}(\Gamma)$, the category of unbounded chain complexes of Γ -comodules, by inverting the homology isomorphisms. This is not the right thing to do to form $\text{Stable}(\Gamma)$. To see this, note that there is a model structure on $\text{Ch}(\Gamma)$ whose homotopy category is the derived category, as there is on $\text{Ch}(\mathcal{A})$ for any Grothendieck

category \mathcal{A} [Bek00]. The cofibrations in this model structure are the monomorphisms, the fibrations are the epimorphisms with DG-injective kernel, and the weak equivalences are the homology isomorphisms. Here a complex X is **DG-injective** if each X_n is injective and every map from an exact complex to X is chain homotopic to 0. In particular, bounded above complexes of injectives are DG-injective, from which it follows that

$$\mathcal{D}(\Gamma)(A, A)_* = \text{Ext}_\Gamma^*(A, A).$$

Here we are thinking of A as a complex concentrated in degree 0. It is the analog of the sphere S , since it is the unit of \wedge . Now, there are often non-nilpotent elements in $\text{Ext}_\Gamma^i(A, A)$ for $i > 0$. For example, the well-known element β_1 is non-nilpotent in $\text{Ext}_{BP_*BP}^*(BP_*, BP_*)$ for $p > 2$. Since β_1 corresponds to a self-map of BP_* in $\mathcal{D}(BP_*BP)$, we can form $\beta_1^{-1}BP_*$. But this complex has trivial homology, so is 0 in $\mathcal{D}(BP_*BP)$ even though β_1 is not nilpotent. This is not good for several reasons; we should be able to see $\beta_1^{-1}BP_*$ because it is an important object, and the fact that we can't also implies that A is not a small object of $\mathcal{D}(BP_*BP)$.

We should be inverting homotopy isomorphisms, not homology isomorphisms. To do this, we need to define the homotopy groups.

Definition 3.1. Suppose (A, Γ) is an Adams Hopf algebroid, $X \in \text{Ch}(\Gamma)$, and P is a dualizable Γ -comodule. We define the **homotopy groups of X with coefficients in P** by

$$\pi_n^P(X) = H_{-n}(\Gamma\text{-comod}(P, LA \wedge X)),$$

where LA is the cobar resolution of A . We define a map f of complexes to be a **homotopy isomorphism** if $\pi_n^P(f)$ is an isomorphism for all n and all dualizable comodules P . The **stable homotopy category of Γ** , $\text{Stable}(\Gamma)$ is defined to be the category obtained from $\text{Ch}(\Gamma)$ by inverting the homotopy isomorphisms.

The reason for the sign in the definition of π_n^P is so that

$$\pi_n^P(M) \cong \text{Ext}_\Gamma^n(P, M)$$

for a comodule M . Homotopy groups and homotopy isomorphisms satisfy the expected properties [Hov02a].

Proposition 3.2. *Suppose (A, Γ) is an Adams Hopf algebroid, and P is a dualizable comodule.*

- (a) *A short exact sequence of complexes induces a long exact sequence in the homotopy groups $\pi_*^P(-)$. Note that the boundary map raises dimension by one, because of the sign in the definition of $\pi_n^P(-)$.*
- (b) *Homotopy groups commute with filtered colimits, so homotopy isomorphisms are closed under filtered colimits.*
- (c) *Every chain homotopy equivalence is a homotopy isomorphism, and every homotopy isomorphism is a homology isomorphism.*
- (d) *The natural map $X \rightarrow LA \wedge X$ is a homotopy isomorphism for all X .*

To understand $\text{Stable}(\Gamma)$, and for that matter to even see that it is a category at all, we need a model structure on $\text{Ch}(\Gamma)$ in which the weak equivalences are the homotopy isomorphisms. This is the main goal of [Hov02a], where the following theorem is proved.

Theorem 3.3. *Suppose (A, Γ) is an Adams Hopf algebroid. Then there is a proper symmetric monoidal model structure on $Ch(\Gamma)$ in which the weak equivalences are the homotopy isomorphisms. Furthermore, if X is cofibrant, then $X \wedge (-)$ preserves homotopy isomorphisms.*

We call this model structure the **homotopy model structure**. The cofibrations in the homotopy model structure are dimensionwise split monomorphisms with cofibrant cokernel. If X is cofibrant, then each X_n is projective over A . More precisely, if X is cofibrant, then X is a retract of a complex Y that admits a filtration Y^i such that each map $Y^i \rightarrow Y^{i+1}$ is a dimensionwise split monomorphism and the quotient Y^{i+1}/Y^i is a complex of relatively projective comodules with trivial differential. Here a comodule is **relatively projective** if it is a retract of a direct sum of dualizable comodules.

A characterization of the fibrations is given in [Hov02a]. Fibrations are of course surjective. Every complex of relative injectives is fibrant, and every fibrant complex is equivalent in a precise sense to a complex of relative injectives. A fibrant replacement of X is given by $LB \wedge X$.

The following theorem is also proved in [Hov02a].

Theorem 3.4. *The homotopy model structure is natural, in the sense that a map $\Phi: (A, \Gamma) \rightarrow (B, \Sigma)$ induces a left Quillen functor $\Phi_*: Ch(\Gamma) \rightarrow Ch(\Sigma)$ of the homotopy model structures. Furthermore, if Φ is a weak equivalence, then Φ_* is a strong Quillen equivalence, in the sense that both Φ_* and Φ^* preserve and reflect homotopy isomorphisms.*

Global properties of $Stable(\Gamma)$. We now establish some of the essential properties of $Stable(\Gamma)$. At this point, we begin to use some of the standard notational conventions of ordinary stable homotopy. Thus, we will begin using S for the image of A in $Stable(\Gamma)$, thinking of it as analogous to the usual zero-sphere. Similarly, we will sometimes use $[X, Y]_*$ for graded maps in $Stable(\Gamma)$. In practice, Γ is usually graded and so $Stable(\Gamma)$ is bigraded. However, the internal suspension in the category of Γ -comodules is usually not relevant, so we tend to omit it from the notation.

Theorem 3.5. *Suppose (A, Γ) is an Adams Hopf algebroid. The category $Stable(\Gamma)$ is a closed symmetric monoidal triangulated category. The dualizable comodules form a set of small, dualizable, weak generators for $Stable(\Gamma)$.*

This theorem is really a corollary of Theorem 3.3 and general facts about model categories. It is proved in [Hov02a]; another way to say it is that $Stable(\Gamma)$ is a unital algebraic stable homotopy category in the sense of [HPS97].

One drawback of $Stable(\Gamma)$ is that it is not in general monogenic. That is, A and its suspensions are not generally enough to generate the whole category. This is unavoidable even for Hopf algebras. Indeed, if G is a finite group and k is a field, the stable homotopy category of the Hopf algebra of functions from G to k is closely related to the stable module category much studied in modular representation theory [Ben98], as explained in [HPS97]. If G is a p -group, the stable module category is monogenic, but not in general.

However, one certainly expects $Stable(BP_*BP)$ and $Stable(E(n)_*E(n))$ to be monogenic, so we need a condition on (A, Γ) that will ensure that $Stable(\Gamma)$ is monogenic. Recall that a full subcategory \mathcal{D} of an abelian category is called **thick**

if it is closed under retracts and, whenever two out of three terms in a short exact sequence are in \mathcal{D} , so is the third.

Proposition 3.6. *Suppose (A, Γ) is an Adams Hopf algebroid, and every dualizable comodule is in the thick subcategory generated by A . Then $\text{Stable}(\Gamma)$ is monogenic.*

This proposition is proved in [Hov02a]. To apply it, we note that the filtration theorem for E_*E -comodules, part (b) of Theorem 2.11, implies that every finitely presented E_*E -comodule is in the thick subcategory generated by E_* when E is a commutative ring spectrum that is Landweber exact over BP_* . Thus we get the following corollary.

Corollary 3.7. *Suppose E is a commutative ring spectrum that is Landweber exact over BP . Then $\text{Stable}(E_*E)$ is monogenic in the bigraded sense. In particular, a map f in $\text{Ch}(E_*E)$ is a homotopy isomorphism if and only if $\pi_{n,k}(f) = \pi_n^{s^k E_*}(f)$ is an isomorphism for all n and k .*

The bigrading arises because we can suspend a complex X either internally, by suspending each graded comodule X_n , or externally by suspending the complex X .

We would like to understand the relation between a comodule M and its image in $\text{Stable}(\Gamma)$. The following proposition is proved in [Hov02a].

Proposition 3.8. *Suppose (A, Γ) is an Adams Hopf algebroid.*

- (a) *A short exact sequence of comodules, or even complexes, gives rise to a cofiber sequence in $\text{Stable}(\Gamma)$.*
- (b) *If M is in the thick subcategory generated by A , then M is a small object of $\text{Stable}(\Gamma)$.*
- (c) *If M and N are comodules, then there is a natural map*

$$\text{Ext}_{\Gamma}^k(M, N) \rightarrow \text{Stable}(\Gamma)(M, N)^k$$

that is an isomorphism for M in the thick subcategory generated by A .

Part (b) is not actually proved in [Hov02a], but follows immediately from part (a). In particular, of course, we have

$$\text{Stable}(\Gamma)(A, A)^* \cong \pi_*^A(A) \cong \text{Ext}_{\Gamma}^*(A, A).$$

This is the stable homotopy of the sphere in $\text{Stable}(\Gamma)$.

The following point is also valuable.

Proposition 3.9. *Suppose E is a commutative ring spectrum that is Landweber exact over BP . Then $\text{Stable}(E_*E)$ is a Brown category, so that every homology functor is representable.*

This proposition follows from Theorem 4.1.5 of [HPS97]. Indeed, we can assume $E = E(n)$ or BP , and then one can easily check using the cobar resolution that $\text{Ext}_{E_*E}^{**}(E_*, E_*)$ is countable.

Ordinary homology. We now describe ordinary homology in $\text{Stable}(\Gamma)$. Since homotopy isomorphisms are in particular homology isomorphisms, the ordinary homology of a chain complex X is a homology theory on $\text{Stable}(\Gamma)$.

Proposition 3.10. *Let (A, Γ) be an Adams Hopf algebroid. Ordinary homology is represented on $\text{Stable}(\Gamma)$ by Γ itself, as usual thought of as a complex concentrated in degree 0.*

Because of this proposition, we will sometimes denote Γ by H .

Proof. For a complex X , we have

$$\Gamma_*(X) \cong \pi_*(Q\Gamma \wedge QX),$$

where Q denotes cofibrant replacement. Since $QX \wedge (-)$ preserves homotopy isomorphisms by Theorem 3.3, we have

$$\pi_*(Q\Gamma \wedge QX) \cong \pi_*(\Gamma \wedge QX).$$

Since $\Gamma \wedge QX$ is already fibrant, as it is a complex of relative injectives, we have

$$\pi_*(\Gamma \wedge QX) \cong \text{Ch}(\Gamma)(A, \Gamma \wedge QX) / \sim,$$

where \sim denotes the chain homotopy relation. This is in turn isomorphic to

$$\text{Ch}(A)(A, QX) / \sim \cong H_*(QX) \cong H_*X$$

by adjointness. □

Note that the Hopf algebroid (H_*, H_*H) associated to homology is isomorphic to (A, Γ) itself, concentrated in degree 0. Thus H_*X is naturally a graded (A, Γ) -comodule, which is bigraded in case (A, Γ) is graded. We get an Adams-Novikov spectral sequence based on H whose E_2 -term is

$$E_2^{s,t} \cong \text{Ext}_\Gamma^s(A, H_t X),$$

which in good cases will converge to π_*X . In particular, if $X = S$, this spectral sequence is concentrated in degrees $(s, 0)$, and so collapses and converges to

$$\pi_*S \cong \text{Ext}_\Gamma^*(A, A).$$

Thus, if we take $(A, \Gamma) = (BP_*, BP_*BP)$, we have built a stable homotopy category in which the usual Adams-Novikov spectral sequence collapses.

Note that ordinary cohomology is somewhat complicated. This is actually already true in the derived category $\mathcal{D}(A)$. Indeed, in $\mathcal{D}(A)$ we have

$$H^*(X) \cong \text{Ch}(A)(QX, S^0 A)^*$$

and there is no really convenient interpretation of these groups. Similarly, in $\text{Ch}(\Gamma)$, we have

$$H^*(X) \cong \text{Ch}(\Gamma)(QX, S^0 \Gamma)^* \cong \text{Ch}(A)(QX, S^0 A)^*.$$

The ordinary derived category of Γ , obtained by inverting the homology isomorphisms, is the Bousfield localization of $\text{Stable}(\Gamma)$ with respect to H . As we have said before, this is a non-trivial localization. Indeed, suppose x is a non-nilpotent class in $\text{Ext}^s(A, A)$ with $s > 0$. Then x corresponds to a self-map $S^{-s} \rightarrow S$, which is necessarily 0 on homology. Hence the telescope $x^{-1}S$ will have no homology, but will be nonzero. In particular, if $(A, \Gamma) = (BP_*, BP_*BP)$, there are non-nilpotent classes in Ext . For example, $\bar{\alpha}_1$ is non-nilpotent when $p = 2$ by [Rav86, Theorem 4.4.37].

Homology with coefficients. We now consider ordinary homology with coefficients in an A -module B .

Proposition 3.11. *Let (A, Γ) be an Adams Hopf algebroid, and let B be an A -module. Then $\Gamma \otimes_A B$ represents the homology theory HB on $\text{Stable}(\Gamma)$ defined by*

$$(HB)_*(X) \cong H_*(B \otimes_A QX).$$

If B is Landweber exact over A , then

$$(HB)_*(X) \cong H_*(B \otimes_A X) \cong B \otimes_A H_*X.$$

In particular, in this case the Hopf algebroid (HB_*, HB_*HB) is isomorphic to (B, Γ_B) concentrated in degree 0.

Proof. For an object X of $\text{Stable}(\Gamma)$, we have

$$(HB)_*(X) \cong \pi_*(\Gamma \otimes_A B) \wedge QX \cong H_*(B \otimes_A QX),$$

where we have used Lemma 1.4 to manipulate the tensor product. In particular, $HB_*(S) \cong B$ concentrated in degree 0. If B is Landweber exact over A , then $B \otimes_A (-)$ will preserve homology isomorphisms of complexes of comodules, so

$$(HB)_*(X) \cong H_*(B \otimes_A X) \cong B \otimes_A H_*X.$$

In particular,

$$(HB)_*(HB) \cong B \otimes_A \Gamma \otimes_A B \cong \Gamma_B.$$

Thus $(HB)_*X$ is naturally a graded comodule over (B, Γ_B) , when B is Landweber exact over A . \square

Thus we get theories $HE(n)$ when $(A, \Gamma) = (BP_*, BP_*BP)$ and $B = E(n)_*$.

The Adams-Novikov spectral sequence based on HB when B is Landweber exact will then have E_2 -term

$$E_2^{s,t} \cong \text{Ext}_{\Gamma_B}^s(B \otimes_A H_tX, B \otimes_A H_tY).$$

In particular, when $X = Y = S$, this spectral sequence must collapse, since the E_2 -term is concentrated where $t = 0$. However, it is not entirely clear to what it converges. The obvious guess is $\pi_*L_{HB}S$, where L_{HB} denotes Bousfield localization with respect to HB . Bousfield's convergence results [Bou79] should be re-examined to see if they apply in a more general setting to answer this question.

Note that if B is an A -algebra that is also a field, then HB will be a field object of $\text{Stable}(\Gamma)$. In particular, if \mathfrak{p} is a prime ideal in A with residue field $k_{\mathfrak{p}}$, then we can form $Hk_{\mathfrak{p}}$. If we apply this to the case $(A, \Gamma) = (BP_*, BP_*BP)$, we get field spectra $HK(n)$ corresponding to the Morava K -theories, but we also get many other field spectra, including $H\mathbb{F}_p$ corresponding to the prime ideal I_{∞} . Note that the objects $HK(n)$ do not detect nilpotence in $\text{Stable}(BP_*BP)$, since there are non-nilpotent self-maps of S that are zero on homology with any coefficients.

4. LANDWEBER EXACTNESS AND THE STABLE HOMOTOPY CATEGORY

Recall that in Section 2 we showed that the abelian category of $E(n)_*E(n)$ -comodules is a localization of the abelian category of BP_*BP -comodules. In Section 3, we introduced stable homotopy categories of $E(n)_*E(n)$ and BP_*BP -comodules. It is therefore natural to conjecture that $\text{Stable}(E(n)_*E(n))$ is a Bousfield localization of $\text{Stable}(BP_*BP)$. The goal of this section is to prove this conjecture, thereby proving Theorem C.

The functor Φ_* . Throughout this section, we let $(A, \Gamma) = (BP_*, BP_*BP)$, $B = E(n)_*$, and we let $\Phi: (A, \Gamma) \rightarrow (B, \Gamma_B)$ be the induced map of Hopf algebroids.

The map of Hopf algebroids Φ induces a functor

$$\Phi_*: \Gamma\text{-comod} \rightarrow \Gamma_B\text{-comod}$$

and a left Quillen functor

$$\Phi_*: \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

by Theorem 3.4. To prove Theorem C, we must show that Φ_* induces a Quillen equivalence upon suitably localizing $\text{Ch}(\Gamma)$. The object of the present section is to prove the following theorem.

Theorem 4.1. *The functor $\Phi_*: \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$ preserves weak equivalences. Its right adjoint Φ^* reflects weak equivalences.*

We prove this theorem in a series of propositions.

Proposition 4.2. *Let \mathcal{D} denote the class of all $X \in \text{Ch}(\Gamma)$ such that the map $\Phi_*QX \rightarrow \Phi_*X$ is a weak equivalence in $\text{Ch}(\Gamma_B)$, where Q is a cofibrant replacement functor in $\text{Ch}(\Gamma)$. Then \mathcal{D} is a thick subcategory.*

Proof. Note that \mathcal{D} is obviously closed under retracts. To see that \mathcal{D} is thick, suppose we have a short exact sequence

$$X' \rightarrow X \rightarrow X''$$

in $\text{Ch}(\Gamma)$ such that two out of three terms are in \mathcal{D} . By Proposition 3.8(a), this is a cofiber sequence in $\text{Stable}(\Gamma)$. Since Φ_*Q is the total left derived functor of the left Quillen functor Φ_* , we conclude that

$$\Phi_*QX' \rightarrow \Phi_*QX \rightarrow \Phi_*QX''$$

is a cofiber sequence in $\text{Stable}(\Gamma_B)$. On the other hand, because Φ_* is exact, the sequence

$$\Phi_*X' \rightarrow \Phi_*X \rightarrow \Phi_*X''$$

is a short exact sequence in $\text{Ch}(\Gamma_B)$, and hence, applying Proposition 3.8(a) again, is also a cofiber sequence in $\text{Stable}(\Gamma_B)$. There is a map from the first of these cofiber sequences to the second, and by assumption it is an isomorphism on two out of three terms. Since $\text{Stable}(\Gamma_B)$ is a triangulated category, we conclude that it is also an isomorphism on the third term, and so \mathcal{D} is thick. \square

Our next goal is to show that \mathcal{D} is closed under filtered colimits. For this we need to recall some standard model category theory. Suppose \mathcal{I} is a small category, and \mathcal{M} is a cofibrantly generated model category, such as $\text{Ch}(\Gamma)$. Then there is a cofibrantly generated model category structure on the diagram category $\mathcal{M}^{\mathcal{I}}$ [Hir03, Theorem 12.7.1] in which the weak equivalences and fibrations are taken objectwise. Furthermore, the cofibrations in $\mathcal{M}^{\mathcal{I}}$ are in particular objectwise cofibrations [Hir03, Proposition 12.7.3].

Proposition 4.3. *The class \mathcal{D} of Proposition 4.2 is closed under filtered colimits.*

Proof. Suppose $F: \mathcal{I} \rightarrow \text{Ch}(\Gamma)$ is a functor from a filtered small category \mathcal{I} such that $F(i) \in \mathcal{D}$ for all $i \in \mathcal{I}$. We must show that $\text{colim} F(i) \in \mathcal{D}$. Let QF be a cofibrant replacement of F in the model category on $\text{Ch}(\Gamma)^{\mathcal{I}}$ discussed prior to this proposition. Because the constant diagram functor obviously preserves fibrations

and trivial fibrations, the colimit is a left Quillen functor [Hir03, Theorem 12.7.9]. Hence $\operatorname{colim} QF$ is cofibrant. Furthermore, each map $QF(i) \rightarrow F(i)$ is a homotopy isomorphism, and so, since homotopy commutes with filtered colimits, we conclude that the map $\operatorname{colim} QF \rightarrow \operatorname{colim} F$ is a weak equivalence. Therefore, $\operatorname{colim} QF$ is a cofibrant replacement of $\operatorname{colim} F$. To show that $\operatorname{colim} F \in \mathcal{D}$, then, we need only show that the map $\Phi_*(\operatorname{colim} QF) \rightarrow \Phi_*(\operatorname{colim} F)$ is a homotopy isomorphism. Since Φ_* itself commutes with colimits, this is equivalent to showing that the map $\operatorname{colim} \Phi_* QF \rightarrow \operatorname{colim} \Phi_* F$ is a homotopy isomorphism. Since QF is cofibrant, and cofibrations of diagrams are in particular objectwise cofibrations, we conclude that $QF(i)$ is a cofibrant replacement for $F(i)$ for all $i \in \mathcal{I}$. Since $F(i) \in \mathcal{D}$, then, each map $\Phi_* QF(i) \rightarrow \Phi_* F(i)$ is a homotopy isomorphism. Hence, again using the fact that homotopy commutes with filtered colimits, $\operatorname{colim} \Phi_* QF \rightarrow \operatorname{colim} \Phi_* F$ is a homotopy isomorphism, so $\operatorname{colim} F \in \mathcal{D}$. \square

We now know that \mathcal{D} is a thick subcategory that is closed under filtered colimits and (obviously) contains all the cofibrant objects of $\operatorname{Ch}(\Gamma)$. This should mean that it has to be all of $\operatorname{Ch}(\Gamma)$, and that is what we now prove.

Proposition 4.4. *If $X \in \operatorname{Ch}(\Gamma)$, then the map $\Phi_* QX \rightarrow \Phi_* X$ is a weak equivalence.*

Proof. The proposition is just saying that the class \mathcal{D} of Propositions 4.2 and 4.3 is all of $\operatorname{Ch}(\Gamma)$. We prove this in three steps. We first show that the complexes $S^n M$ are in \mathcal{D} , where M is a finitely presented Γ -comodule and $S^n M$ denotes the complex whose only non-zero entry is M in degree n . We then show that all finitely presented complexes are in \mathcal{D} , and finally, we show that every complex is a filtered colimit of finitely presented complexes, so is in \mathcal{D} by Proposition 4.3.

For the first step, it is clear that $S^n A$ is in \mathcal{D} since it is cofibrant. The collection of all M such that $S^n M$ is in \mathcal{D} is a thick subcategory by Proposition 4.2; by induction, therefore, it contains A/I_k for all k . The Landweber filtration theorem then implies that it contains all finitely presented M .

Now suppose X is a finitely presented complex. For the purposes of the present proof, we take this to mean that X_n is finitely presented for all n and 0 for almost all n ; this is in fact equivalent to X being a finitely presented object of $\operatorname{Ch}(\Gamma)$ in the categorical sense. We easily prove by induction on the number of non-zero entries in X that $X \in \mathcal{D}$. Indeed, the base case of one non-zero entry is handled in the preceding paragraph. For the induction step, let X' be the subcomplex of X obtained by removing the non-zero entry in the largest possible degree. Then $X' \in \mathcal{D}$ by the induction hypothesis, and the quotient $X/X' \in \mathcal{D}$ by the preceding paragraph. Since \mathcal{D} is thick, $X \in \mathcal{D}$.

Now suppose X is an arbitrary complex. Let \mathcal{F}/X denote the category of all maps $F \rightarrow X$, where F is a finitely presented complex. This is easily seen to have a small skeleton and to be a filtered category. There is an obvious inclusion functor $i: \mathcal{F}/X \rightarrow \operatorname{Ch}(\Gamma)$, and an obvious map

$$f: \operatorname{colim} i \rightarrow X.$$

We claim that f is an isomorphism. To see that f is surjective, choose $x \in X_n$. Since the comodule X_n is a filtered colimit of finitely presented comodules, there is a finitely presented comodule F and a map $F \rightarrow X_n$ whose image contains x . This gives a map of complexes $D^n F \rightarrow X$ whose image contains x , and so f is

surjective. To see that f is injective, suppose $j: F \rightarrow X$ is an object of \mathcal{F}/X and $x \in F_n$ has $yx = 0$. Let K denote the kernel of the map j , so that $x \in K_n$. Now K may not be finitely presented, but at least there is a map $F' \rightarrow K_n$ from a finitely presented comodule whose image contains x . This corresponds to a map $D^n F' \rightarrow K$ of complexes, which induces an object $F/D^n F' \rightarrow X$ of \mathcal{F}/X and a map $F \rightarrow F/D^n F'$ in \mathcal{F}/X that sends x to 0. Thus x is 0 in $\text{colim } i$ and so f is injective. \square

We can now give the proof of Theorem 4.1.

Proof of Theorem 4.1. Suppose $f: X \rightarrow Y$ is a weak equivalence in $\text{Ch}(\Gamma)$. Then Qf is a weak equivalence between cofibrant objects, so $\Phi_* Qf$ is a weak equivalence since Φ_* is a left Quillen functor. But we have a commutative square

$$\begin{array}{ccc} \Phi_* QX & \xrightarrow{\Phi_* Qf} & \Phi_* QY \\ \downarrow & & \downarrow \\ \Phi_* X & \xrightarrow{\Phi_* f} & \Phi_* Y \end{array}$$

where the vertical maps are weak equivalences, by Proposition 4.4. Hence $\Phi_* f$ is a weak equivalence as well.

To prove the second part of Theorem 4.1, suppose f is a map in $\text{Ch}(\Gamma_B)$ such that $\Phi^* f$ is a weak equivalence in $\text{Ch}(\Gamma)$. Then $\Phi_* \Phi^* f$ is a weak equivalence in $\text{Ch}(\Gamma)$ by what we have just proved. But f is naturally isomorphic to $\Phi_* \Phi^* f$ by Theorem 2.2. \square

Localization. We have just seen that $\Phi_*: \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$ preserves weak equivalences. But of course it does not reflect weak equivalences, since $\Phi_*(A/I_{n+1}) = 0$. We dealt with this problem already in the abelian category world by localizing Γ -comod so as to force $0 \rightarrow A/I_{n+1}$ to be an isomorphism. We now want to do the same thing for chain complexes.

More precisely, we define $L_n^f \text{Ch}(\Gamma)$ to be the category $\text{Ch}(\Gamma)$ equipped with the model structure that is the Bousfield localization of the homotopy model structure with respect to the maps $0 \rightarrow s^k A/I_{n+1}$ for all k , where $s^k A/I_{n+1}$ denotes the complex which is A/I_{n+1} in degree k and 0 elsewhere. Recall from [Hir03] that this means that a left Quillen functor

$$F: \text{Ch}(\Gamma) \rightarrow \mathcal{M}$$

defines a left Quillen functor

$$F: L_n^f \text{Ch}(\Gamma) \rightarrow \mathcal{M}$$

if and only if $0 \rightarrow (LF)(s^k A/I_{n+1})$ is an isomorphism in $\text{ho } \mathcal{M}$, where LF denotes the total left derived functor of F . The homotopy category of $L_n^f \text{Ch}(\Gamma)$ is the finite localization $L_n^f \text{Stable}(\Gamma)$ in the sense of Miller [Mil92] of $\text{Stable}(\Gamma)$ away from A/I_{n+1} . The total left derived functor of the identity, thought of as a functor from $\text{Ch}(\Gamma)$ to $L_n^f \text{Ch}(\Gamma)$, is the finite localization functor L_n^f on $\text{Stable}(\Gamma)$.

Proposition 4.5. *The Quillen functor*

$$\Phi_*: \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

induces a left Quillen functor

$$\Phi_*: L_n^f \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B).$$

Proof. We need to show that $(L\Phi_*)(A/I_{n+1}) = 0$. In light of Proposition 4.4, $(L\Phi_*)X$ is naturally isomorphic to Φ_*X for any $X \in \text{Ch}(\Gamma)$. Thus

$$(L\Phi_*)(A/I_{n+1}) \cong \Phi_*(A/I_{n+1}) = 0. \quad \square$$

Note that Φ_* still preserves weak equivalences when thought as a functor from $L_n^f \text{Ch}(\Gamma)$,

Proposition 4.6. *The Quillen functor*

$$\Phi_*: L_n^f \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

preserves weak equivalences, and its right adjoint Φ^ reflects weak equivalences.*

Proof. Suppose $f: X \rightarrow Y$ is a weak equivalence. Factor $f = pi$, where i is a trivial cofibration and p is a trivial fibration. Then Φ_*i is a weak equivalence since Φ_* is a left Quillen functor. On the other hand, since the trivial fibrations do not change under Bousfield localization, p is a weak equivalence in $\text{Ch}(\Gamma)$. Thus Theorem 4.1 implies that Φ_*p is a weak equivalence. Hence $\Phi_*f = (\Phi_*p)(\Phi_*i)$ is a weak equivalence, as required.

The proof that Φ^* reflects weak equivalences is the same as the proof of the corresponding part of Theorem 4.1. \square

To prove Theorem C, we will show that

$$\Phi_*: L_n^f \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

is a Quillen equivalence. To do so, we will use the following lemma, which is proved in [Hov99, Corollary 1.3.16].

Lemma 4.7. *Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a left Quillen functor of model categories, with right adjoint U . Then F is a Quillen equivalence if and only if the following two conditions hold.*

- (a) *U reflects weak equivalences between fibrant objects.*
- (b) *The map $X \rightarrow URFX$ is a weak equivalence for all cofibrant X in \mathcal{C} , where R denotes fibrant replacement and the map is induced by the unit of the adjunction.*

We have already seen in Proposition 4.6 that Φ^* reflects all weak equivalences. We point out that there is a much simpler proof that Φ^* reflects weak equivalences between fibrant objects; if X is fibrant in $\text{Ch}(\Gamma_B)$, then adjointness implies that $\pi_*(\Phi^*X) \cong \pi_*X$.

The other condition of Lemma 4.7 is harder to check. Here are the main points of the argument.

- (a) We first show, using the fact that Φ^* preserves filtered colimits, that it suffices to show that $A \rightarrow \Phi^*(LB)$ is an L_n^f -equivalence, where LB denotes the cobar resolution of B .
- (b) A Bousfield class argument that shows that it suffices to prove that

$$v_k^{-1}A/I_k \rightarrow v_k^{-1}A/I_k \wedge Q\Phi^*(LB)$$

is a homotopy isomorphism for all $0 \leq k \leq n$, where Q denotes cofibrant replacement.

- (c) We show that, although $\Phi^*(LB)$ is not cofibrant, it is still nice enough that it suffices to check that

$$v_k^{-1}A/I_k \rightarrow v_k^{-1}A/I_k \wedge \Phi^*(LB)$$

is a homotopy isomorphism.

- (d) It was proved in [Hov02b] that the Hopf algebroid $(v_k^{-1}A/I_k, v_k^{-1}\Gamma/I_k)$ is weakly equivalent to $(v_k^{-1}B/I_k, v_k^{-1}\Gamma_B/I_k)$. Hence it suffices to show that

$$v_k^{-1}B/I_k \rightarrow v_k^{-1}B/I_k \wedge \Phi_*\Phi^*(LB) \cong v_k^{-1}B/I_k \wedge LB$$

is a homotopy isomorphism, and this is clear.

We now fill in the details of this argument, beginning with Step (a).

Proposition 4.8. *The Quillen functor*

$$\Phi_*: L_n^f \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

is a Quillen equivalence if and only if the map

$$A \rightarrow \Phi^*(LB)$$

is an L_n^f -equivalence.

Recall that LB denotes the cobar resolution of B as a Γ_B -comodule.

Before proving this proposition, we need a lemma.

Lemma 4.9. *The total right derived functor*

$$R\Phi^*: \text{Stable}(\Gamma_B) \rightarrow L_n^f \text{Stable}(\Gamma)$$

preserves coproducts.

Proof. First note that because L_n^f is a finite localization, it is in particular smashing [Mil92]. Thus the coproduct in $L_n^f \text{Stable}(\Gamma)$ is the same as the coproduct in $\text{Stable}(\Gamma)$. Also note that a fibrant replacement in $\text{Stable}(\Gamma_B)$ is given by $LB \wedge (-)$, which clearly preserves coproducts. Hence, it suffices to show that

$$\Phi^*: \Gamma_B\text{-comod} \rightarrow \Gamma\text{-comod}$$

preserves coproducts. Since Φ^* certainly preserves finite coproducts, and any coproduct is a filtered colimit of finite coproducts, it suffices to show that Φ^* preserves all filtered colimits. This follows from the fact that $L_n = \Phi^*\Phi_*$ preserves filtered colimits (Theorem 2.8); more details can be found in [HS03b]. \square

Proof of Proposition 4.8. Lemma 4.7 tells us that if Φ_* is a Quillen equivalence, then $A \rightarrow \Phi^*R\Phi_*A$ must be a weak equivalence in $L_n^f \text{Ch}(\Gamma)$. Since $\Phi_*A = B$, and since LB is a fibrant replacement for B in $\text{Ch}(\Gamma_B)$, we conclude that $A \rightarrow \Phi^*(LB)$ is an L_n^f -equivalence.

Conversely, suppose $A \rightarrow \Phi^*(LB)$ is an L_n^f -equivalence. By Lemma 4.7 and Proposition 4.6, it suffices to show that $X \rightarrow \Phi^*R\Phi_*X$ is an L_n^f -equivalence for all cofibrant X . This is equivalent to proving that

$$X \xrightarrow{\eta_X} (R\Phi^*)(L\Phi_*)X$$

is an isomorphism in $L_n^f \text{Stable}(\Gamma)$ for all X , where $R\Phi^*$ denotes the total right derived functor of Φ^* and $L\Phi_*$ denotes the total left derived functor of Φ_* . Let \mathcal{D} denote the full subcategory of $L_n^f \text{Stable}(\Gamma)$ of those X such that η_X is an isomorphism. By hypothesis, \mathcal{D} contains $L_n^f A$. Since $L\Phi_*$ and $R\Phi^*$ both preserve exact triangles, \mathcal{D} is a thick subcategory. As a left adjoint, $L\Phi_*$ preserves coproducts,

and Lemma 4.9 assures us that $R\Phi^*$ also preserves coproducts. Thus \mathcal{D} is a localizing subcategory. In any monogenic stable homotopy category, the only localizing subcategory that contains the unit is the whole category. \square

We are now reduced to showing that $A \rightarrow \Phi^*(LB)$ is an L_n^f -equivalence. The theory of Bousfield classes gives us the following lemma.

Lemma 4.10. *A map f of cofibrant objects in $Ch(\Gamma)$ is an L_n^f -equivalence if and only if $v_k^{-1}A/I_k \wedge f$ is a weak equivalence for all k with $0 \leq k \leq n$.*

Proof. As usual, let $\langle X \rangle$ denote the Bousfield class of X in $\text{Stable}(\Gamma)$, thought of as the collection of all Y in $\text{Stable}(\Gamma)$ such that $X \wedge Y = 0$. The cofiber sequences

$$A/I_k \xrightarrow{v_k} A/I_k \rightarrow A/I_{k+1}$$

imply that

$$\langle A/I_k \rangle = \langle v_k^{-1}A/I_k \rangle \vee \langle A/I_{k+1} \rangle,$$

by [Rav84, Lemma 1.34]. Thus, we find

$$\langle A \rangle = \bigvee_{k=0}^n \langle v_k^{-1}A/I_k \rangle \vee \langle A/I_{n+1} \rangle.$$

As in the usual stable homotopy category, this implies that L_n^f is localization with respect to $\bigoplus_{k=0}^n v_k^{-1}A/I_k$.

It follows that f is an L_n^f -equivalence if and only if $v_k^{-1}A/I_k \wedge^L f$ is a weak equivalence for all k with $0 \leq k \leq n$, where \wedge^L denotes the total left derived functor of \wedge . Recall from Theorem 3.3 that if X is cofibrant, then $X \wedge (-)$ preserves homotopy isomorphisms. It follows that, if X is cofibrant, $(-) \wedge^L X \cong (-) \wedge X$ in $\text{Stable}(\Gamma)$. Hence, if f is a map of cofibrant objects, then f is an L_n^f -equivalence if and only if $v_k^{-1}A/I_k \wedge f$ is a weak equivalence for all k with $0 \leq k \leq n$. \square

By combining Proposition 4.8 with Lemma 4.10, we get the following corollary, which is Step (b) of the argument on page 22.

Corollary 4.11. *The Quillen functor*

$$\Phi_*: L_n^f Ch(\Gamma) \rightarrow Ch(\Gamma_B)$$

is a Quillen equivalence if and only if the map

$$v_k^{-1}A/I_k \rightarrow v_k^{-1}A/I_k \wedge Q\Phi^*(LB)$$

is a homotopy isomorphism for all $0 \leq k \leq n$, where Q denotes a cofibrant replacement functor in $Ch(\Gamma)$.

To accomplish Step (c) of the argument on page 22, we need to know something about $\Phi^*(LB)$.

Lemma 4.12. *We have*

$$\text{Tor}_A^j(A/I_k, \Phi^*(LB)_m) = 0$$

for all $j > 0$, $k \geq 0$, and $m \in \mathbb{Z}$.

Proof. Recall that

$$(LB)_{-m} = \Gamma_B \otimes_B \overline{\Gamma}_B^{\otimes_B m}$$

for $m \geq 0$, and is 0 otherwise. Here $\overline{\Gamma}_B$ is the cokernel of the left unit $\eta_L: B \rightarrow \Gamma_B$. Since this map is split as a map of B -modules, $\overline{\Gamma}_B$ is a flat B -module. Therefore $\overline{\Gamma}_B^{\otimes_B m}$ is also a flat B -module, and hence a filtered colimit of projective B -modules.

Now, we have

$$\Phi^*(LB)_{-m} = \Gamma \otimes_A \overline{\Gamma}_B^{\otimes_B m}.$$

Since $\mathrm{Tor}_A^j(A/I_k, -)$ commutes with filtered colimits, it suffices to show that

$$\mathrm{Tor}_A^j(A/I_k, \Gamma \otimes_A M) = 0$$

for all $j > 0$, all k , and all projective B -modules M . But then we can easily reduce to the case $M = B$, so we must show that

$$\mathrm{Tor}_A^j(A/I_k, \Gamma \otimes_A B) = 0$$

for all $j > 0$ and all k . We prove this by induction on k , using the exact sequences

$$0 \rightarrow A/I_k \xrightarrow{v_k} A/I_k \rightarrow A/I_{k+1} \rightarrow 0.$$

We are reduced to showing that v_k is not a zero-divisor on $(\Gamma \otimes_A B)/I_k$. Since I_k is invariant, this is the same as showing that v_k is not a zero-divisor on $\Gamma \otimes_A (B/I_k)$. Since v_k is itself invariant modulo I_k and Γ is flat over A , this is in turn equivalent to showing that v_k is not a zero-divisor on B/I_k . This follows because B is Landweber exact. \square

With this lemma in hand, we can carry out Step (c) of our argument.

Proposition 4.13. *The map*

$$v_k^{-1}A/I_k \wedge Q\Phi^*(LB) \rightarrow v_k^{-1}A/I_k \wedge \Phi^*(LB)$$

is a homotopy isomorphism in $Ch(\Gamma)$. In particular,

$$\Phi_*: L_n^f Ch(\Gamma) \rightarrow Ch(\Gamma_B)$$

is a Quillen equivalence if and only if

$$v_k^{-1}A/I_k \rightarrow v_k^{-1}A/I_k \wedge \Phi^*(LB)$$

is a homotopy isomorphism for all $0 \leq k \leq n$.

Proof. Let q denote the map

$$q: Q\Phi^*(LB) \rightarrow \Phi^*(LB).$$

Because homotopy commutes with filtered colimits, it suffices to show that $A/I_k \wedge q$ is a homotopy isomorphism for all k .

Now, q is a trivial fibration in the homotopy model structure, so we have a short exact sequence of complexes

$$0 \rightarrow K \rightarrow Q\Phi^*(LB) \xrightarrow{q} \Phi^*(LB) \rightarrow 0.$$

The long exact sequence in homotopy implies that $\pi_t^A(K) = 0$ for all n . Lemma 4.12 implies that we have a short exact sequence

$$0 \rightarrow A/I_k \wedge K \rightarrow A/I_k \wedge Q\Phi^*(LB) \rightarrow A/I_k \wedge \Phi^*(LB) \rightarrow 0$$

for all k . The long exact sequence in homotopy implies that we need only check that $\pi_t^A(A/I_k \wedge K) = 0$ for all n .

To see this, note that the short exact sequence defining K realizes K_m as the first syzygy of $\Phi^*(LB)_m$, since $(Q\Phi^*(LB))_m$ is projective over A . Hence Lemma 4.12 implies that

$$\mathrm{Tor}_A^j(A/I_k, K_m) = 0$$

for all $j > 0$. Hence we have short exact sequences

$$0 \rightarrow A/I_k \wedge K \rightarrow A/I_k \wedge K \rightarrow A/I_{k+1} \wedge K \rightarrow 0$$

for all k . The long exact sequence in homotopy and induction on k now complete the proof. \square

The final step of the argument on page 22 requires us to know more about the Hopf algebroids $(v_k^{-1}A/I_k, v_k^{-1}\Gamma/I_k)$ and $(v_k^{-1}B/I_k, v_k^{-1}\Gamma_B/I_k)$.

Lemma 4.14. *Let (C, Σ) denote $(v_k^{-1}A/I_k, v_k^{-1}\Gamma/I_k)$, and let (C_B, Σ_B) denote $(v_k^{-1}B/I_k, v_k^{-1}\Gamma_B/I_k)$. Then:*

- (a) *Both (C, Σ) and (C_B, Σ_B) are Adams Hopf algebroids;*
- (b) *The stable homotopy categories $\mathrm{Stable}(\Sigma)$ and $\mathrm{Stable}(\Sigma_B)$ are monogenic;*
- (c) *If $0 \leq k \leq n$, Φ induces a weak equivalence of Hopf algebroids*

$$\Phi: (C, \Sigma) \rightarrow (C_B, \Sigma_B).$$

Proof. For part (a), we know that Γ is a filtered colimit of comodules that are finitely generated projective A -modules. By tensoring with $v_k^{-1}A/I_k$, we see that Σ is a filtered colimit of comodules that are finitely generated projective C -modules, and so (C, Σ) is an Adams Hopf algebroid. Similarly, (C_B, Σ_B) is an Adams Hopf algebroid.

For part (b), the proof is again the same for (C, Σ) and (C_B, Σ_B) , so we concentrate on (C, Σ) . We will use Proposition 3.6, so we need to show that every dualizable Σ -comodule is in the thick subcategory generated by C . We will do this by showing that every finitely presented Σ -comodule has a Landweber filtration. To do so, we will use the Hopf algebroid $(v_k^{-1}A, v_k^{-1}\Gamma v_k^{-1})$, obtained by inverting v_k and $\eta_R v_k$. A Σ -comodule M is just a $(v_k^{-1}\Gamma v_k^{-1})$ -comodule on which I_k acts trivially. Since I_k is finitely generated, M is finitely presented if and only if it is finitely presented as a $v_k^{-1}\Gamma v_k^{-1}$ -comodule. Since $v_k^{-1}A$ is Landweber exact of height k , Theorem 2.11 gives us a Landweber filtration of M as a $v_k^{-1}\Gamma v_k^{-1}$ -comodule in which each filtration quotient is isomorphic to $v_k^{-1}A/I_j$ for some $j \leq k$. Since M is killed by I_k , in fact each filtration quotient must be isomorphic to $v_k^{-1}A/I_k$, giving us our Landweber filtration of M as a Σ -comodule.

Part (c) is a special case of Theorem E of [Hov02b]. \square

Lemma 4.14 allows us to carry out the final step of our argument.

Proposition 4.15. *Suppose f is a map in $\mathrm{Ch}(\Gamma)$, and $0 \leq k \leq n$. Then $v_k^{-1}A/I_k \wedge f$ is a homotopy isomorphism in $\mathrm{Ch}(\Gamma)$ if and only if $v_k^{-1}B/I_k \wedge \Phi_* f$ is a homotopy isomorphism in $\mathrm{Ch}(\Gamma_B)$.*

Proof. Let $C = v_k^{-1}A/I_k$ and let $\Sigma = v_k^{-1}\Gamma/I_k$, as in Lemma 4.14. The category of Σ -comodules is just the full subcategory of Γ -comodules on which I_k acts trivially and v_k acts invertibly. This follows from the fact that I_k is invariant and v_k is primitive modulo I_k . Thus, if X is a complex in $\mathrm{Ch}(\Sigma)$, we can also think of X as a complex in $\mathrm{Ch}(\Gamma)$. As such, X has homotopy groups $\pi_*^C(X)$ and $\pi_*^A(X)$. We claim that these are naturally isomorphic. Indeed, one can easily check that LC , the

cobar complex of (C, Σ) , is just $v_k^{-1}LA/I_k$. Hence $LC \wedge X$ is naturally isomorphic to $LA \wedge X$. From this, one can easily check the desired isomorphism.

Therefore, using parts (a) and (b) of Lemma 4.14, we conclude that $v_k^{-1}A/I_k \wedge f$ is a homotopy isomorphism in $\text{Ch}(\Gamma)$ if and only if it is a weak equivalence in $\text{Ch}(\Sigma)$. Similarly, let $C_B = v_k^{-1}B/I_k$ and $\Sigma_B = v_k^{-1}\Gamma_B/I_k$. We find that $v_k^{-1}B/I_k \wedge \Phi_*f$ is a homotopy isomorphism in $\text{Ch}(\Gamma_B)$ if and only if it is a weak equivalence in $\text{Ch}(\Sigma_B)$.

Now, use part (c) of Lemma 4.14 and Theorem 3.4 to conclude that $v_k^{-1}A/I_k \wedge f$ is a weak equivalence in $\text{Ch}(\Sigma)$ if and only if

$$v_k^{-1}B/I_k \otimes_{v_k^{-1}A/I_k} (v_k^{-1}A/I_k \wedge f) \cong v_k^{-1}B/I_k \wedge \Phi_*f$$

is a weak equivalence in $\text{Ch}(\Sigma_B)$. \square

We can now complete the proof of Theorem C, which we first restate in stronger form.

Theorem 4.16. *The Quillen functor*

$$\Phi_*: L_n^f \text{Ch}(\Gamma) \rightarrow \text{Ch}(\Gamma_B)$$

is a Quillen equivalence. Furthermore, Φ_ and its right adjoint Φ^* preserve and reflect weak equivalences.*

Proof. Proposition 4.13 and Proposition 4.15 imply that, to check that Φ_* is a Quillen equivalence, we only need to check that the map

$$v_k^{-1}B/I_k \rightarrow v_k^{-1}B/I_k \wedge \Phi_*\Phi^*(LB)$$

is a homotopy isomorphism for $0 \leq k \leq n$. But $\Phi_*\Phi^*$ is naturally isomorphic to the identity functor by Theorem 2.2. Hence we need only check that the map

$$v_k^{-1}B/I_k \rightarrow v_k^{-1}B/I_k \wedge LB$$

is a homotopy isomorphism, which follows from Proposition 3.2(d).

We have already seen that Φ_* preserves weak equivalences and that Φ^* reflects them in Proposition 4.6. Suppose f is a map in $\text{Ch}(\Gamma)$ such that Φ_*f is a weak equivalence. Since Φ_* preserves weak equivalences, it follows that Φ_*Qf is a weak equivalence. But, since Φ_* is a Quillen equivalence, it must reflect weak equivalences between cofibrant objects by [Hov99, Corollary 1.3.16]. Hence Qf is a weak equivalence, and so f is a weak equivalence. This proves that Φ_* reflects weak equivalences.

Now suppose g is a weak equivalence in $\text{Ch}(\Gamma_B)$. By Theorem 2.2, g is naturally isomorphic to $\Phi_*\Phi^*g$. Since Φ_* reflects weak equivalences, we conclude that Φ^*g is a weak equivalence. \square

We point out that it is possible to further localize the Quillen equivalence in Theorem 4.16 to obtain a Quillen equivalence

$$\Phi_*: L_n \text{Ch}(\Gamma) \rightarrow L_n \text{Ch}(\Gamma_B)$$

where L_n is Bousfield localization with respect to $HE(n)$ is the first case, and ordinary homology H in the second.

The change of rings theorem. In this final part, we show how our work implies the Miller-Ravenel change of rings theorem. To begin with, here is our generic change of rings theorem. Recall our notational conventions: $(A, \Gamma) = (BP_*, BP_*BP)$, $B = E(n)_*$, and $\Gamma_B = E(n)_*E(n)$.

Theorem 4.17. *Suppose $X \in Ch(\Gamma)$ and Y is an L_n^f -local object of $Stable(\Gamma)$. Then*

$$Stable(\Gamma)(X, Y)^* \cong Stable(\Gamma_B)(\Phi_*X, \Phi_*Y)^*.$$

Proof. First of all, since Φ_* preserves weak equivalences, we have

$$Stable(\Gamma_B)(\Phi_*X, \Phi_*Y) \cong Stable(\Gamma_B)(\Phi_*QX, \Phi_*QY).$$

Also, by Theorem 4.16, we have

$$Stable(\Gamma_B)(\Phi_*QX, \Phi_*QY) \cong (L_n^fStable(\Gamma))(X, Y).$$

Since Y is already L_n^f -local, we have

$$(L_n^fStable(\Gamma))(X, Y) \cong Stable(\Gamma)(X, Y),$$

as required. \square

We claim that this corollary captures the Miller-Ravenel change of rings theorem. To see this, we need the following lemma.

Lemma 4.18. *Suppose N is a Γ -comodule with $L_nN = N$ and $L_n^iN = 0$ for $i > 0$. Then N is an L_n^f -local object of $Stable(\Gamma)$.*

Given this lemma, we immediately get the following corollary, which is Theorem D and more like the usual change of rings theorems.

Corollary 4.19. *Suppose M is a finitely presented BP_*BP -comodule, and N is a BP_*BP -comodule such that $L_nN = N$ and $L_n^iN = 0$ for all $i > 0$. Then*

$$\text{Ext}_{BP_*BP}^{**}(M, N) \cong \text{Ext}_{E(n)_*E(n)}(E(n)_* \otimes_{BP_*} M, E(n)_* \otimes_{BP_*} N).$$

This corollary includes both the Miller-Ravenel change of rings theorem [MR77], by taking N with $N = v_n^{-1}N$, and the change of rings theorem of the author and Sadofsky [HS99], by taking N with $v_j^{-1}N = N$ for some $j \leq n$. Here we are using Theorem 2.8(e) to verify the hypothesis of Corollary 4.19.

Proof. Simply apply Theorem 4.17, using Lemma 4.18 to see that N is L_n^f -local, and Proposition 3.8(c) to identify the groups in question as Ext groups. \square

We still owe the reader a proof of Lemma 4.18, which, incidentally, is presumably a special case of a spectral sequence that computes $H_*(L_n^fX)$ for $X \in Stable(\Gamma)$ from the derived functors $L_n^iH_*X$, in analogy to the spectral sequence of Theorem 2.9. The converse of Lemma 4.18 is true as well, though we do not need it.

Proof of Lemma 4.18. Note that, by definition, N is L_n^f -local if and only if

$$Stable(\Gamma)(A/I_{n+1}, N)^* = 0.$$

This is equivalent to

$$\text{Ext}_{\Gamma}^i(A/I_{n+1}, N) = 0$$

for all i , by Proposition 3.8(c). Now $\text{Ext}_{\Gamma}^i(A/I_{n+1}, N) = 0$ for $i = 0, 1$ if and only if $L_nN = N$, by Theorem 2.8(a). Suppose in addition that $L_n^iN = 0$ for $i > 0$.

We claim that we can find cosyzygies C_j of N in the category of Γ -comodules such that $L_n C_j = C_j$ and $L_n^i C_j = 0$ for $i > 0$. If we assume this, then for $i > 0$ we have

$$\mathrm{Ext}_{\Gamma}^i(A/I_{n+1}, N) \cong \mathrm{Ext}_{\Gamma}^1(A/I_{n+1}, C_{i-1}) = 0.$$

We now construct the cosyzygies C_j by induction on j , taking $C_0 = N$. Suppose we have constructed C_j . Since $L_n C_j = C_j$, C_j has no v_n -torsion. It follows that the injective hull, or indeed any essential extension of C_j , has no v_n -torsion. Therefore, we can find an short exact sequence

$$0 \rightarrow C_j \rightarrow I_j \rightarrow C_{j+1} \rightarrow 0$$

of Γ -comodules where I_j is an injective comodule with no v_n -torsion. In particular, $L_n I_j = I_j$ by Theorem 2.8(a). If we apply L_n to this sequence, we find that $L_n C_{j+1} = C_{j+1}$ and $L_n^i C_{j+1} = 0$ for $i > 0$, completing the induction step. \square

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